J. Bernstein - Nearby & Vanishing Cycles Berkeley 1/02

X alg variety $\rightsquigarrow D_{const}(X)$ 6 operations

Ψ - functor of nearby cycles : $\begin{matrix} X \\ \downarrow f \\ \mathbb{A}^1 \end{matrix}$ f a fibration/smooth over $\mathbb{A}^1$ -

X nonsingular ~~projective~~

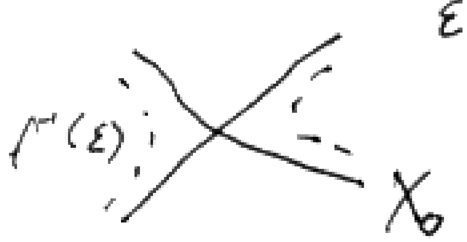
$\varepsilon \rightsquigarrow X_\varepsilon = f^{-1}(\varepsilon)$

$\varepsilon \neq 0 \Rightarrow$ nonsingular variety

$H^*(X_\varepsilon)$ form local system in ε for $\varepsilon \neq 0$: topological fibration $\varepsilon \neq 0$.

$\varepsilon = 0$ can define $H^i(X_\varepsilon)$ for $\varepsilon > 0$ small - (canonically) indep of ε. get a vector space which we call space of nearby cycles

$H^i_+ = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} H^i(X_\varepsilon) = \lim H^i(X_\varepsilon, \mathbb{C}_{X_\varepsilon})$



$X = \mathbb{A}^2$ $f(x,y) = xy$

Claim: $\exists$ a complex of sheaves $\psi(\mathbb{C}_X)$ on X_0 s.t. for any open subset $U \subset X$, $H^i(X_0 \cap U, \psi(\mathbb{C}_X)) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} H^i(X_\varepsilon \cap U, \mathbb{C}_X)$

- localize this picture.

$\psi(\mathbb{C}_X)$ is canonically $\mathbb{C}_{X_0}$ <u>outside</u> of singularities.

$F \in D_{const}(X^*)$ $X^* = f^{-1}(\mathbb{A}^*)$ $(\mathbb{A}^* = \mathbb{A}^1 - 0)$ $\begin{matrix} X_0 \subset X \\ \downarrow \quad \downarrow \\ 0 \in \mathbb{A}^1 \end{matrix}$

...

Deligne: Y top space, F local system on Y - what is a fiber of F?

e.g. can take fiber at a point $y \in Y$.

Other "fibers": suppose B contractible top space, $\nu: B \to Y$

$\nu^* F$ - local system on B - canonically trivial

- so can define fiber of F at B, $F/_B$.

i.e. point is just an example of a contractible space

$Y = \mathbb{C}^* \Rightarrow$ can take  cone

Fix tangent vector at $0 \Rightarrow$ define fiber at this tangent vector

So if we fix tangent vector at $0 \in \mathbb{A}^1$ $\Rightarrow$ define nearby cycles for this tangent vector.

Take a universal cover $\tilde{\mathbb{C}} \longrightarrow \mathbb{C}^*$

$$\begin{array}{c} \tilde{X} = \tilde{\mathbb{C}} \times_{\mathbb{C}^*} X^* \\ p \downarrow \\ X \end{array}$$

$F^\bullet$ sheaf / complex on X^*

$p^* F^\bullet$ on $\tilde{X}$, $\rightsquigarrow$ $j_* p_* p^* F^\bullet$ on X

restrict to X_0 $\quad \psi(F^\bullet) = i^*(j_* p_* p^* F^\bullet)$

$$\psi : D^b_{const}(X^*) \longrightarrow D_{const}(X_0)$$

Consider case $X = \mathbb{C}$: $\quad p_* p^* \mathbb{C}_X$ sheaf on $\mathbb{C}^*$, fiber at point is product over all its inverse images in $\tilde{\mathbb{C}}$.

$\rightsquigarrow$ Monodromy operator $\mu \circlearrowright p_* p^* \mathbb{C}_X$ from. $\quad \mu : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ over $\mathbb{C}^*$

but highly nonconstructible sheaf! - nonetheless restricting to X_0 get a constructible sheaf...

- we've used nonalgebraic map $\tilde{\mathbb{C}} \longrightarrow \mathbb{C}^*$

F local system on $\mathbb{C}^*$, $F = \mathbb{C}_\mu$ one-dim with monodromy μ.

F trivializes on $\tilde{\mathbb{C}}$, apply $(jp)_*$ $\quad jp : \tilde{\mathbb{C}} \to \mathbb{C}$

$i^*(jp)_* p^* F = \Gamma((jp)^{-1} U, p^* F)$ $\quad U$ nbhd of 0

$U = \odot$ $\quad \Rightarrow (jp)^{-1}(U)$ is halfplane, contractible, just get 1-dim space of sections

$\Rightarrow$ 1-dim space $\mathbb{C}$ with action of monodromy μ.

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k any algebraically closed field.

$$\begin{array}{ccc} X_0 \subset & X & \supset X^* \\ & \downarrow & \\ 0 \in & \mathbb{A}^1 & \supset \mathbb{A}^* \end{array}$$

ψ: $D_{const\ \ell\text{-adic}}\, X^* \longrightarrow D_{const\ \ell\text{-adic}}\, X_0$

$\mathbb{A}^1 = \operatorname{Spec} k[t]$, $\mathbb{A}^* \Rightarrow \operatorname{Spec} k((t))$ punctured disc

$\tilde{\mathbb{C}} \Rightarrow \operatorname{Spec} \overline{k((t))}$ algebraic closure - must be fixed.

$$\begin{array}{ccccccc} X_0 & \xhookrightarrow{i} & X & \xhookleftarrow{j} & X^* & \longleftarrow & \tilde{X} \\ & & & & \downarrow & & \\ & & & & \mathbb{A}^* & \longleftarrow & \tilde{\mathbb{A}} \end{array}$$

$i^* j_* p_* p^* F$

Poincaré groupoid: "choose" all universal covers: form a groupoid

$\mathcal{P}$ — Poincaré groupoid of Spec $k((t))$ parametrizes functors ψ

$$\psi : \mathcal{P} \to \mathrm{Funct}(D_{const}\, X^{\circ}, D_{const}\, X_0)$$

$X \downarrow \mathbb{A}^1$ $\quad H^*(X_t, \mathbb{C}_X)$ has monodromy root of unity:

write X over $\mathbb{Z}$, reduce mod p — almost all p

same answer $\Rightarrow$ X over $\mathbb{F}_p$, $\mathbb{F}_p((t))$ —

no wild ramification almost all p $\Rightarrow$ $\hat{\mathbb{Z}}$ acts as ~~does~~ Frobenius
μ

$Fr\,\mu = \mu^p\, Fr \Rightarrow \mu$ quasi-unipotent.

Beilinson: ψ for $\mathcal{D}$-modules, $X/\mathbb{C}$.

R-H $\quad D_{const}(X) \longleftrightarrow D_{RS}(\mathcal{D}_X)$

How to describe ψ directly on $D_{RS}(\mathcal{D}_X)$

$\psi(F) \circlearrowleft \mu$, μ has fin many eigenvalues ($\psi(F)$ constructible...

$\psi(F) = \bigoplus \psi(F)_{\mu=\mu_i} \supset \psi(F)_{\mu=1} = \psi_{un}(F)$ unipotent part

- from $\mu=1$ part can recover others:

$\psi_{un}(F \otimes \mathbb{C}_{\mu_i^{-1}}) = \psi_{\mu=\mu_i}(F)$ $\quad \mathbb{C}_{\mu_i^{-1}}$ local system on $\mathbb{A}^{1*}$

[Consider unipotent local system on $\mathbb{C}^*$, given by unipotent matrix in Jordan form $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$. Γ_1.]

F, $\mathcal{D}$-mod on X^*, holonomic $\quad j_*(F \cdot x^{\lambda})$ $\quad \mathcal{D}$-module on X depending on λ — i.e. $\mathcal{D}[\lambda]$-module (x^λ $\mathcal{D}$-mod on $\mathbb{A}^1$, all hol...

Inside here find $\mathcal{D}$-submodules, which are everything outside X_0

let $j_!(F \cdot x^{\lambda})$ be ~~the~~ minimal submodule of $j_*(F x^\lambda)$ which coincides with $j_*(F x^\lambda)$ off X_0:

i.e. $x^N \cdot$ any section will land in the submodule, $N \gg 0$

- so $\mathbb{Z}$ just [problem: using λ not holonomic so $j_!$ not quite defined

Take generators of $j_* F(x^{\lambda})$, generate module by multiplying by x^N,

$j_*(F x^{\lambda}) / j_!(F x^{\lambda})$ D-module supported on X_0,

with operator of mult by λ

Claim: This is $\psi_{unip}(F)[1]$ & λ is monodromy.

Consequence $\psi[1]$: Perv $\to$ Perv preserves perversity!
(can prove directly...)

F perverse sheaf on X^*, $\mathcal{H} = \psi(F)$, $\mu : \mathcal{H} \to \mathcal{H}$

restrict to unipotent part

$N = \mu - 1 : \mathcal{H}_{unip}$ nilpotent operator

Deligne: M object in abelian category + nilpotent operator $N : M \to M$

$\Rightarrow \exists!$ filtration $\{F_i M\}$ s.t. $N F_i \subset F_{i-2}$,

$N^i : Gr_{i}(M) \xrightarrow{\sim} gr_{-i}(M)$

Let F be pure of weight zero & perverse.

$\psi_{unip}(F)$ will be perverse but not pure: in fact

$Gr_i \, \psi_{unip}(F)$ has weight $\pm i$ - unipotent filtration

coincides with weight filtration.

Canonical sl_2 action on Gr

$\mathbb{D}$ interchanges sign & $\mathbb{D}\psi = \psi\mathbb{D}$, N wt -2

$\rightsquigarrow$ property of weight filtration.