MSRI
3/02

R. Bezrukavnikov - Sheaves on affine flags & modular representations of Langlands dual Lie algebra

G simple algebraic group / $\mathbb{C}$
$\mathcal{G}r$ = affine Grassmannian $\quad \mathcal{F}\ell$ = affine flag variety
- ind-schemes / $\mathbb{C}$

$$\mathcal{G}r(\mathbb{C}) = G(\mathbb{C}((t))) / G(\mathbb{C}[[t]])$$
$$\mathcal{F}\ell(\mathbb{C}) = G(\mathbb{C}((t))) / I \qquad \text{Iwahori: } \begin{matrix} I & \subset & G(\mathbb{C}[[t]]) \\ \downarrow & & \downarrow \\ B & \subset & G \end{matrix}$$

Study categories of perverse sheaves on $\mathcal{G}r$, $\mathcal{F}\ell$. $\quad k = \bar{k}$ field of coefficients

spherical sheaves:
$$P_{sph} = Perv^{k}_{G(\mathbb{C}[[t]])}(\mathcal{G}r) \qquad \text{equivariant perverse sheaves}$$
$$P_I = Perv^{k}_{I}(\mathcal{F}\ell)$$

Geometric Satake Theorem (Drinfeld, Ginzburg, Mirkovic-Vilonen)
P_{sph} is a tensor category under convolution & is equivalent to $Rep({}^LG)$ category of reps of Langlands dual group / k

(M.-V.: general coefficient rings k)

We'll define a category P related to P_I for char $k = \ell \neq 0$,
& a category R related to $Rep({}^L\mathfrak{g})$ Lie algebra reps s.t. $P \simeq R$.
(P, R abelian categories)

Structure of argument: indirect - compute both sides ...
Argument will identify
$$D^b(P) \underset{\varphi_1}{\simeq} D^b(Coh^{{}^LG}(T^* {}^LG/{}^LB))$$
$$D^b(R) \underset{\varphi_2}{\simeq} \qquad \text{equivariant coherent sheaves}$$

The composition ϕ of φ_1, φ_2 will have better properties than φ_1, φ_2. $\quad \varphi_1$ variant of work jt. with S. Arkhipov
φ_2 joint with Mirkovic-Rumynin

(I) Definition of P. $\quad$ {isom classes of irred objects in P_I} $\leftrightarrow W$
$IC_w \leftarrow\!\shortmid w$

W = affine Weyl group
IC_w = intersection cohomology of I-orbit through w.

Let $W_f \subset W$ be finite Weyl group

${}^fW$ = set of minimal reps of cosets in W/W_f =
$\{ w \in W \mid \ell(w_f \cdot w) \geq \ell(w) \ \forall w_f \in W_f \}$

$P :=$ Serre quotient of P_I with ${}^fW$ as set of irreducible objects: $P = P_I / \langle IC_w \mid w \notin {}^fW \rangle$

Remark 1. P is categorical (categorification) counterpart of antispherical module for the affine Hecke algebra $\mathcal{H}$

$K^0(P)$ has natural W action & as such is isomorphic to $\mathbb{Z}[W] \otimes_{\mathbb{Z}[W_f]} \text{sign}$ = antispherical module (induced rep of sign)

2. P has a Whittaker realization: $P \simeq$ category of Iwahori-Whittaker perverse sheaves (Bez.-Arkhipov)

Let $\mathcal{N} \subset {}^L\mathfrak{g}$ be the nilpotent cone, $\tilde{\mathcal{N}} = T^*({}^LG/{}^LB)$ Springer resolution of $\mathcal{N}$.

Theorem 1 (ℓ = char k = 0 or large) $\rightsquigarrow$ probably just > Coxeter # of ${}^L\mathfrak{g}$

$$D^b(P) \simeq D^b(\mathrm{Coh}^{{}^LG}(\tilde{\mathcal{N}}))$$

... isom on K-groups this induces
$K^0(\mathrm{Coh}^{{}^LG}(\tilde{\mathcal{N}})) \simeq$ antispherical is well known - e.g. used in Kazhdan-Lusztig, see Chriss-Ginzburg.

(II) Definition of R : $[\text{char } k = \ell > 2h({}^LG) - 2 > 0]$

Start with category $\mathrm{Rep}\,{}^L\mathfrak{g} = \mathrm{Rep}\, U\,{}^L\mathfrak{g}$

Recall that the center Z of $U\,{}^L\mathfrak{g}$ contains two parts:

$Z \supset Z_{HC} = U({}^L\mathfrak{g})^{{}^LG} = \mathrm{Sym}(\mathfrak{h})^{W_f}$ $\quad \mathfrak{h}$ = Cartan

$\cup$ (Harish-Chandra center)

$Z_{Fr} = \langle x^\ell - x^{[\ell]} : x \in {}^L\mathfrak{g} \rangle \simeq \mathrm{Sym}\, \mathfrak{g}^{(1)}$ $\quad x^{[\ell]}$ = restricted power

(Frobenius center) $\rightsquigarrow$ Frob. twist

$\mathrm{Spec}\, Z = \mathfrak{g}^{*(1)} \times_{(\mathfrak{h}^*/W)^{(1)}} \mathfrak{h}^*/W$ $\quad$ where $\mathfrak{h}^*/W \longrightarrow \mathfrak{h}^*/W$ is Artin-Schreier

For $\lambda \in \mathfrak{h}^*$ let $\psi_\lambda : Z_{HC} \longrightarrow k$ be the corresponding character. For $\lambda \in \mathfrak{h}^*(\mathbb{F}_p)$, $Z \underset{Z_{HC}}{\otimes} \psi_\lambda \simeq \mathcal{O}(\mathcal{N})$

eg ψ_0 = character of trivial module
$\psi_{-\rho}$ = " " Steinberg module

$R_0 = Rep_{f.g.}(U({}^L\mathfrak{g}) \underset{Z_{HC}}{\otimes} \psi_0)$ finitely generated reps

<u>Theorem</u> (B-Mirkovic-Rumynin) $\exists$ an Azumaya algebra $\mathcal{A}$ on $\tilde{\mathcal{N}}$ & an equivalence btw. $D^b(Coh_{\mathcal{A}}(\tilde{\mathcal{N}}))$ coherent sheaves of $\mathcal{A}$-modules

$$\wr\!\wr \quad D^b(R_0)$$

<u>Rmk</u> $\mathcal{A}$ = (a version of) differential operators in characteristic p ("crystalline differs") — variant of Beilinson-Bernstein localization.

Let R_1 be the category of Harish-Chandra modules for the pair $({}^L\mathfrak{g} \oplus {}^L\mathfrak{g}, {}^LG)$ ... $\rightarrow$ diagonal where Z_{HC} acts by $\psi_0 \otimes \psi_{-\rho}$ regular char $\otimes$ most singular char.

<u>Remark</u> The action of $U({}^L\mathfrak{g} \oplus {}^L\mathfrak{g})$ on $M \in R_1$ factors through $\bar{U} = (U({}^L\mathfrak{g}) \underset{Z_{Fr}}{\otimes} U({}^L\mathfrak{g})) \underset{Z_{HC} \otimes Z_{HC}}{\otimes} (\psi_0 \otimes \psi_{-\rho})$

$\bar{Z}$ = Image of $Z \otimes Z$ in $\bar{U}$ can be described as $\bar{Z} \simeq \mathcal{O}(\mathcal{N})$, & for $x \in \mathcal{N}$ the reduction $\bar{U}_x = \bar{U} \underset{\bar{Z}}{\otimes} x$ is Morita equivalent to $U({}^L\mathfrak{g}) \underset{Z}{\otimes} (\psi_0, x)$

-- so for f.d reps doesn't matter if we work with U or $\bar{U}$.

<u>Theorem</u> $D^b(R_1) \simeq D^b(Coh^{{}^LG}(\tilde{\mathcal{N}}))$

$D^b(R_1)$ carries a Serre-type duality & a self-dual t-structure

Duality is defined by $M \mapsto RHom_{\bar{U}}(M, \bar{U})$

"coherent perverse t-structure" : defined as follows

Def $M \in D^b(R_1)$ ies in $D^{p, \leq 0}$ if for any ${}^L G$-orbit $X \subset N$, we have $M \otimes^L_{\mathcal{O}(N) = \bar{Z}} \mathcal{O}(X) \in D^{\leq \frac{1}{2} \operatorname{codim} X}$

Remark Forgetful functor $F: R_1 \longrightarrow Coh^{{}^L G}(N^{(1)})$ ⟶ Fr-twist
(forget all but action of ${}^L G$ & $\bar{Z}$)
F sends $\mathcal{S}$ to Serre duality
& our t-structure to the t-structure of perverse coherent sheaves for middle perversity. (uniquely characterizes t-structure)
(cf math AG ...)

Def R = core of t-structure.

Theorem $D^b(R) \simeq D^b(P)$ sends P to R, $\mathcal{S}$ to Verdier duality
$\quad D^b(Coh^{{}^L G}(\tilde{N}))$

Remark Closely motivated by results & conjectures of Lusztig!
(on characters of nonrestricted representations)

Corollary $\forall \chi \in N$ there is a subquotient (some quotient of abelian subcategory) P_χ of P which is equivalent to the category of H-C modules ($\bar{U}_\chi$, $Z(\chi)$ modules where $Z(\chi) \subset {}^L G$ is stabilizer of χ.

Idea of proof of Theorem 1 $D^b(P) \simeq D^b(Coh^{{}^L G}(\tilde{N}))$:

Basic ingredient 1) [Gaitsgory] Central functor
$Z: \operatorname{Rep} {}^L G = P_{Sph} \longrightarrow P_I$.

2) [Gaitsgory] Z carries a tensor endomorphism : logarithm of monodromy... ⟸ Z defined via nearby cycles.

3) Wakimoto sheaf & filtration of $Z(V)$ by such (in char 0)

Z is a geometric counterpart of the isomorphism of Bernstein-Lusztig description of $Z(\mathcal{H})$ center of affine Hecke with $\mathbb{Z}[\Lambda]^{W_f}$ Λ = coweight lattice

Wakimoto sheaves: geometric counterparts of the embedding $\mathbb{Z}[\Lambda] \hookrightarrow \mathcal{H}$

$\lambda \longmapsto q^{-\ell(\lambda)/2} T_\lambda$ $\quad \lambda \in \Lambda^+$ dominant coweight

$\mathbb{Z}[\Lambda]^{W_f} \subset \mathbb{Z}[\Lambda] \longleftrightarrow$ filtration by Wakimoto sheaves....

$Z(V)$ has a filtration by Wakimoto sheaves

$1,2,3 \xrightarrow[\text{(Tannakian arguments)}]{} $ a tensor functor $F: D^b(Coh^{G}\tilde{N}) \to \mathcal{H}(Fl)$

s.t. $V \otimes \mathcal{O} \longrightarrow Z(V)$

$\mathcal{O}(\lambda) \longmapsto$ Wakimoto sheaf

- check that F composed with projection functor to antispherical quotient is an equivalence.

Two parts - geometric Langlands & char p rep theory & composed equivalence is better for some reason

Replacing char $k = \ell$ by char 0 expect to replace modular reps by reps of quantum group at root of unity.