Luminy
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R. Bezrukavnikov - Geometric Langlands Duality & Positive Characteristic

I. "Analytic" phenomena become algebraic in characteristic p.
Theory of something in char. p has a quasiclassical model.

II. Quantum is close to classical in char p.

I. (with A. Braverman) Geometric Langlands for GL_n over $k = \bar{k}$ of char. p.

Geometric Langlands: (almost) equivalence of derived categories
$$D\text{-mod}(Bun_G) \longleftrightarrow Coh(\mathcal{L}oc_{G^\vee})$$
on moduli stacks of bundles / local systems on a curve C.

In particular for $\mathcal{E}$ a $G^\vee$-local system on C
$\rightsquigarrow$ skyscraper $\mathbb{C}_\mathcal{E} \in Coh(\mathcal{L}oc_{G^\vee})$
$\longmapsto M_\mathcal{E}$ $\mathcal{E}$-automorphic $\mathcal{D}$-module on Bun_G

$\mathcal{D}\text{mod} \simeq \mathcal{D}_{\Omega^{\frac{1}{2}}}\text{-mod}$ $\quad \Omega^{\frac{1}{2}}$ square root of volume forms (critical twisting), so equivalently expect $\mathcal{D}_{\Omega^{\frac{1}{2}}}\text{-mod} \simeq Coh(\mathcal{L}oc)$

In this realization should have $\mathcal{D}_{\Omega^{\frac{1}{2}}}$ itself should correspond to the structure sheaf of $G^\vee$-opers $\mathcal{O}_{Op_{G^\vee}}$.

Now let X be a smooth alg. variety $/k$, of characteristic p. X carries a sheaf of rings $\mathcal{D}_X$
(PD differs / crystalline diffops / enveloping algebra of tangent algebroid)
$$\mathcal{D}_X = \langle \mathcal{O}, Vect \rangle / \text{Weyl algebra relations.}$$

Does not act faithfully on functions: ∂^p acts by 0 on $\mathcal{O}$.

Center: $x_i^p, \partial_i^p \in Z(\mathcal{D}_{\mathbb{A}^n})$. More generally

Prop 1. $Z(\mathcal{D}_X) = \mathcal{O}_{T^*X^{(1)}}$ Frobenius twist
(constants act by p^{th} power: can identify with T^*X for perfect groundfield)

2. $\mathcal{D}_X$ is an Azumaya algebra of rank $p^{2\dim X}$ on $T^*X^{(1)}$ [Mirkovic - Rumynin]

2 ⟿ char p Stone-von Neumann: Weyl relations have unique irred rep once fix action of center.

The Azumaya algebra splits on $X \subset T^*X$
$(\mathcal{D}|_X \simeq \mathrm{End}(Fr_* \mathcal{O}_X))$

The class of this Azumaya algebra comes from the canonical 1-form $p_i d\xi_i$ on T^*X via the map

$$\mathcal{O}^*/(\mathcal{O}^*)^p \xrightarrow{d\log} \Omega^1 \xrightarrow[\text{Cartier} - Id]{} \Omega^1$$

$$\Rightarrow \quad H^0(\Omega^1) \longrightarrow H^1(\mathcal{O}^*/\mathcal{O}^{*p}) \xrightarrow[\text{Kummer}]{} H^2(\mathcal{O}^*,{}^p)$$
$$\shortparallel$$
$$H^2(\mathcal{O}^*)$$

If M is a $\mathcal{D}_X$-module $\Rightarrow$ qcoherent sheaf on $T^*X^{(1)}$. The support of the latter is called the p-support of M.

Theorem (B.-Braverman) If $\mathcal{E}$ is a rank n vector bundle w/ flat connection on C whose p-support is smooth $\Rightarrow \exists!$ irreducible $M_{\mathcal{E}} \in \mathcal{D}_{Bun_G}$-mod

$\Rightarrow$ functor $\mathcal{D}_{Bun_G}$-mod $\longrightarrow$ $\mathrm{Coh}\,\mathrm{Loc}_{s\sim}$

$\twoheadrightarrow \mathcal{D}\text{-mod}/\text{something} \xrightarrow{\sim}$ coherent sheaves on local systems with smooth p-support

Idea of proof: $G = GL_n$. $\mathcal{D}_{Bun_G}$ is
an Azumaya algebra on (an open part of) $T^* Bun_G^{(1)} \simeq (\mathcal{E}, \varphi \in \Omega^1 \otimes ad_{\mathcal{E}})$ Higgs bundles

$S \subset T^*C \times H$ universal spectral curve $\longrightarrow$ $\downarrow \pi$ Hitchin map (invariant polynomials of φ)
$H = \bigoplus_{i=1}^{n} H^0(\Omega^{\otimes i})$

Generic fibers of π are group-schemes ...
Pic of spectral curve (over $H_{sm} \subset H$ locus of smooth spectral curves)

Remark: the Azumaya algebra splits on the geometric fibers of π.
Given a family A of abelian varieties & an Azumaya algebra $\mathcal{A}$ on the total space which splits fiberwise $\Rightarrow$ torsor over the dual family $A^\vee$.
(... torsor of fiberwise splittings of $\mathcal{A}$)

& have twisted Fourier-Mukai equivalence
$\mathcal{A}$-mod $\longleftrightarrow$ $\mathcal{C}oh$(torsor)

On the other side (local systems):
Over $\mathbb{C}$ there are no global algebraic functions on Loc_{GL_n}, though there are analytic ones
e.g. $n=1$ $\quad Loc \xrightarrow{H^0(\Omega^1) = \mathbb{G}_a^g} Pic$ universal additive extension
(analytically though $Loc \simeq (\mathbb{C}^*)^{2g}$)

In characteristic p also have $0 \to Pic^{(1)} \to Loc \to H^0(\Omega^1)^{(1)} \to 0$
$\mathcal{L} \mapsto$ p-curvature ($\Leftarrow$ its graph = p-support)

Likewise $\forall n$ we have $\mathrm{Loc}_{GL_n} \longrightarrow H^{(1)}$
Frobenius twist of Hitchin base

<u>Prop</u> Over $H^{(1)}_{sm}$ this map is a torsor over the Frobenius twist of the Hitchin fiber (universal spectral Picard) & this torsor is dual to the gerbe coming from $\mathcal{D}_{\mathrm{Bun}_G}$.

<u>Lemma</u> Consider oper locus
$$\begin{array}{ccc} & \mathcal{O}p & \subset \mathrm{Loc}_{GL_n} \\ & \varphi \searrow & \downarrow \\ & & H^{(1)} \\ & H \xrightarrow{Fr} & \end{array}$$
Then φ is flat & finite of degree $p^{\dim H}$

(compatible with claim that $\mathrm{Fun}(\mathcal{O}p)$ is a deformation of $\mathrm{Fun}(H)$ .... so this deformation is a deformation <u>over</u> $H^{(1)}$.)

<u>Lemma</u> The equivalence on the generic locus
$$\mathrm{Coh}(\mathrm{Loc}_m) \simeq \mathcal{D}_{\mathrm{Bun}_G, \Omega^{\frac{1}{2}}}\text{-mod}$$
sends $\mathcal{D}_{\Omega^{\frac{1}{2}}}$ to $\mathcal{O}_X$ (restricted to open locus)
for some $X \subset \mathrm{Loc}$ which is finite flat of deg $p^{\dim H}$.

II. <u>Local story</u>

Loose local analog of Hitchin maps $T^*\mathrm{Bun}_G \longrightarrow H$
is $\underbrace{\pi : T^*G/B}_{\tilde{N}} \longrightarrow \mathcal{N}_{\mathfrak{g}}$ Springer map

$D(\mathrm{Coh}\,\tilde{N}) \circlearrowleft B_{aff}$ action of affine braid group of the <u>dual group</u>.
(comes from proof of tamely ramified local Langlands)

similarly for $D(\mathrm{Coh}^G \tilde{N})$, $D(\mathrm{Coh}^{G\times\mathbb{G}_m} \tilde{N})$
& preserving Springer fibers.

On K-groups this action factors through the affine Hecke algebra H_{aff} (or affine Weyl if don't include $\mathbb{G}_m$-equivariance)

In particular $K^0(Coh^{G\times\mathbb{G}_m}(\tilde{N}))$ = antispherical module for H_{aff}. = Iwahori-invariant Whittaker functions on dual group.

<u>Theorem</u> (Arkhipov-B.) $\exists$ canonical equivalence
$$D^b(Coh^G(\tilde{N})) \cong D^b(P_{asph})$$
antispherical (Iwahori-Whittaker) .. sheaves on affine flags for $G^\vee$.

IIB (B - Mirkovic - Rumynin). $k=\bar{k}$ of char $p>h(G)$

Theorem: 1. $R\Gamma$: $D^b(\mathcal{D}\text{-mod}(G/B))$
$$\cong D^b(\Gamma(\mathcal{D})\text{-mod}_{G/B})$$
$$\Gamma(\mathcal{D}) = U\mathfrak{g} \otimes_{(Sym\, \mathfrak{h})^W} k$$

2. The Azumaya algebra $\mathcal{D}_{G/B}$ on $T^*G/B^{(1)}$ splits on the formal neighborhood of Springer fibers.

Claim: the t-structures on $\tilde{N}$ coming from these two pictures are compatible.

. More precisely a subquotient of $P_{antisph}$ is identified with $(A_e\text{-mod})^{Z(e)}$ $e\in N$
where $A_e = A \otimes_{\mathcal{O}(N)} k_e$
& A is a lifting of $U(\mathfrak{g}) \otimes_{Sym\, \mathfrak{h}^W} k$ to char 0.

Why affine Hecke algebra in char p story?

reps in char. 0 controlled by $\mathfrak{h}^* \supset H_\alpha$ comb of constraints.

In char p: $\mathfrak{h}^*(\mathbb{F}_p) \supset H_\alpha$

$$\Lambda/p\Lambda \supset H_{\alpha, pn} \quad : \text{ affine}$$

root systems & Hecke algebra appear from the reduction mod p of root system combinatorics.