

K. Costello - Grover-Witten Theory (Discussion)

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Would like to act by fundamental class of D-M space
to integrate against, to get BV invariant
... ie want $H_*(\mathcal{M}_{g,n}) \rightarrow K(X)^{\otimes n}$
as part of our field theory...

Find replacement:

~~Elgaray~~

$\mathcal{M}(n)$ in boundary, all outgoing
Suppose we have $C^* \mathcal{M}(n) \rightarrow V^{\otimes n}$. (ie don't need)
($V =$ Hochschild chain in our open category) | incoming
compatible with circle actions etc.

Take ~~C^*~~ ($\mathcal{M}(n) / (S')^n \times S_n$)

~~Cycle~~ $I_{t_n}^{U_1}$ = {
~~Submanifold~~ Σ a Riemann surface,
hyperbolic metric, all boundaries have
length $\varepsilon > 0$ & all geodesics
have length $> \varepsilon$

... look away from boundary (short geodesics)

Master equation

$$dH_n + \Delta H_{n+2} = 0 \quad \Delta: \text{gluing operator}$$

if boundary repeat has length ε it
means framings can cut across it ...

S' of possible gluings.

Δ = give other ways of gluing with S' that
... want fundamental class of D-M space, use this as
~~replacement~~

$F(M) = \bigoplus C^* (\mathcal{M}(n) / (S')^n \times S_n)$ commutative algebra
(from disjoint union)

$\Delta : F(M) \rightarrow F(M)$ order two differential operator

$$\Delta^2 = 0 = [d, \Delta], \quad \Delta(c) = 0$$

\Rightarrow BV algebra.

$M_g(n)$ connected genus g surfaces

$$C_*(M_g(n) / (S')^n \times S^n) \rightarrow S_{g,n} :$$

\exists a series $S = \sum \lambda^{2g-2+n} S_{g,n}$, where

$S_{g,n}$ has degree $6g - 6 + 2n$
satisfying $(d + \Delta) e^S$

... unique up to homotopy once we fix

$[S_{0,3}]$ to be class of a point

Build S by induction: start with any 3-holed surface,
 $S_{1,1}$ comes from gluing together with a twist,
get exact this, find something that bounds th.3,
keep building up $[H, M_{1,1} = 0]$
 \Rightarrow exists, $H, M_{1,1} = 0 \Rightarrow$ unique way
which binds this ...]

The chain Ω_n (or a connected version of it) gives
an example of this.

$V =$ Hochschild chain complex

\Rightarrow operator $D: V \rightarrow V$ coming from action of
an anti- : degree 1 operator coming from
b-chain (rotation) in moduli of anti-:

⑤ \langle , \rangle on V coming from gluing, satisfies

$$\langle Dv_1, v_2 \rangle = (-1)^{|v_1|} \langle v_1, Dv_2 \rangle$$

cyclic $V_{S^1} = f^* V[f^{-1}]$ with differential $d + fD$

negative cyclic $V^{S^1} = V[f]$, $d + fD$

periodic cyclic $V_{\text{Tot}} = V((f))$ $d + fD$ so $V_{S^1} = V_{\text{Tot}} / V^{S^1}$

$D^2 = 0$ $[D, \ell] = 0 \Rightarrow D$ is action of $H_*(S')$ on V

& those are homological homotopy fixed points
Borel quotient

Take image $S \in C^*(M(a)) \rightarrow V^{G_a}$

$\Rightarrow S \in (\text{Sym}^* V_{S'})[[T\lambda]]$, satisfies $(d + \Delta)^S = 0$

$\Delta = S^{\text{sing}}$ with Dehn twist (has no mass)

BV operator on $\text{Sym}^* V_{S'}$

$\mathcal{F}(M) \xrightarrow{\Delta} \text{Sym}^* V_{S'} \quad \text{map of } BV \text{ algebras}$

$$\Delta(V, f_1(t), v_2 f_2(t)) = \langle DV, v_2 \rangle R_{v_2} f_1 f_2 e^t.$$

e^t is the Gromov-Witten potential (including descendent $\mapsto t$ variable)

$V_{\text{Total}} = V(f_1), \ell + V$. Give a symplectic form ω_{total}

$$\int (V, f_1(t), v_2 f_2(t)) = \langle V_1, V_2 \rangle R_{v_2} f_1(t) f_2(-t) dt$$

$V_{S'}, V_{S'} \subset V_{\text{Total}}$ Lagrangians

$\text{Sym}^* V_{S'}$ Fact space for symplectic vectors space V_{Total}
 \Rightarrow acted on by Weyl algebra $\text{Weyl}(V_{\text{Total}})$

Lemma $d + \Delta$ on $\text{Sym}^* V_{S'}$ is the only differential compatible with action of $\text{Weyl}(V_{\text{Total}})$:

write $V^{S'} \oplus V_{S'} = V_{\text{Total}}$: generic differential of V_{Total}
 $(\cong$ infinitesimal symplectomorphism) to act on
 the Fact space $\Rightarrow \hat{d}_{S'} = d_{S'} + \Delta$
 $(d_{S'}^2 = d - f D)$

$\int e^s = 0 \Rightarrow$ on homology $[e^s] \in$ Fock space

for $H_x(V_{\text{gate}}) = HP_x$ periodic cycle
homology (with symplectic form of Givental)

Sym $\dagger^* HH_x[\dagger^*]$ \in Fock space

(a prior choice of polarization) : $HP_x = HH_x((f))$
after splitting of Hodge filtration.

(case of a part :

$$H(\mathbb{C}[[t]]) = \mathbb{C}, \quad HP_x(\mathbb{C}) = \mathbb{C}((t))$$

$$\mathcal{L}(f, g) = \text{Res } f(t) g(-t) dt$$

$$HC_x(\mathbb{C}) = \mathbb{C}[[t]] \subseteq \mathbb{C}((t))$$

Let $L \subset \mathbb{C}((t))$ be a constant Lagrangian

$$\text{e.g. } L = t^{-1} \mathbb{C}[t^\pm]$$

$$\text{Fock space} = \text{Sym } L = \text{Sym}^* \dagger^* \mathbb{C}[t^\pm]$$

Potential \mathcal{D}_0 is usual descended potential of a part.

.... Witten-Background independence ..

Potential lies in Fock $(H^3(X))$, which
has a flat connection over moduli space $\mathcal{L} X$
is flat - so lies in background independent space ..

Target to moduli of $\mathcal{O}(X)$ -lines with value b_m ,
is $\frac{1}{13.0} + \frac{1}{4} \sum_i b_i \leq H^3(X)$

get natural 1-form with values in End of Fock space
from Weyl action.

$\underbrace{\mathcal{S}_0}_{\mathcal{S}_0 \in \text{Sym}^* V_S}$ So goes zero part satisfies classical master
eqn, $d + \{S_0, -\}$ gives germ of moduli space of $\frac{NC}{N}$ C^∞