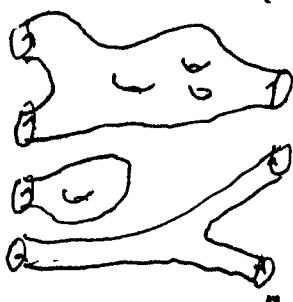


K. Costello - TFTs & Cobordism Categories 12/1/05

Let $M_c(n, m) = \left\{ \begin{array}{l} \text{mod-l. space of } \Sigma \text{ Riemann surface with boundary} \\ \& \text{& an isomorphism } \partial\Sigma \cong S^1 \times \{1, 2, \dots, n+m\} \\ \& \text{1st } n \text{ incoming (outgoing compatible with } \Sigma) \\ \& \& \text{& last } m \text{ outgoing - + condition: every} \\ \& \text{connected component of } \Sigma \text{ carries at least one incoming} \end{array} \right\}$



$M_c(n, m)$ is space of morphisms
in a topological symmetric monoidal category
with objects $\mathbb{Z}_{\geq 0}; M_c$

- $\Sigma_1 \in M_c(n, m), \Sigma_2 \in M_c(r, s)$
 $\Rightarrow \Sigma_2 \circ \Sigma_1 \in M_c(n, r)$: give Outgoing of Σ_1
 to incoming of Σ_2 using parenthesization.
- $\Sigma_1 \in M_c(n, m) \quad \Sigma_2 \in M_c(r, s)$
 $\Rightarrow \Sigma_1 \amalg \Sigma_2 \in M_c(n+r, m+s)$ monoidal struc.
- Units: allow infinitely thin annuli

Field theories (G. Segal)

- 1 Conformal field theory : a symmetric monoidal functor
 $M_c \rightarrow$ Vector spaces
- 2 topological twist : Continuous symmetric monoidal functors to
 topological spaces or spectra.
- Linearized version of 2 : replace all spaces
 by their singular chain complexes.

Def Let C be the category with objects $\mathbb{Z}_{\geq 0}$
& morphisms $C(n, r) = C(M_c(n, r))$
 Singular chains on mod-l. normalized.

This is a differential graded symmetric monoidal category

Def A closed topological conformal field theory is a functor / monoid
 $C \rightarrow$ Chain complexes, compatible wts
 differentials.

i.e. if F is a TFT, for each $n \geq 0$

have chain complex $F(n)$ & chain maps

$$e(n, n) \otimes F(A) \rightarrow F(n)$$

Example: $\Sigma \in M_c(n, n)$ gives a chain map
 $F(\Sigma): F(n) \rightarrow F(n)$

$\Sigma_t : t \in [0, 1]$ 1-parameter family of surfaces

gives $F(\Sigma_t): F(n) \rightarrow F(n)$ degree one
 ...chain homotopy between $F(\Sigma_0) \& F(\Sigma_1)$.

Taking H_0 gives back old defn of TFT.

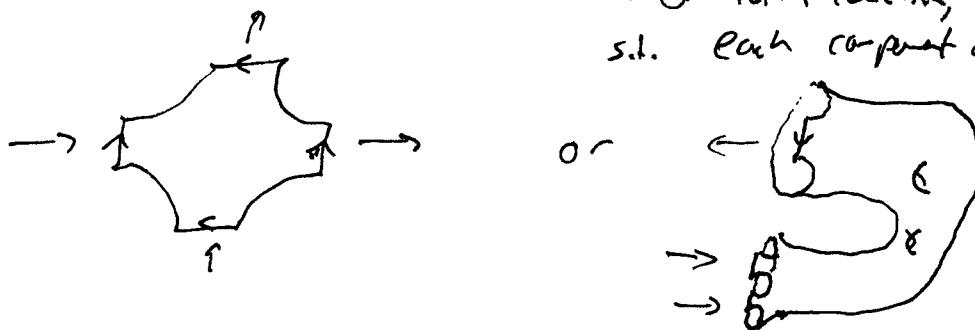
Chain complex/quiver \longleftrightarrow stable Q-homotopy type

(we don't remember the cup product so only get
 stable information).

How to construct these?

1. Do open analog from ribbon graph decompositions
2. Construct closed from open.

Open version: $M_O(n, n) = \{ \sum_i \text{ a R.S. with boundary} \\ \text{ } \cup \text{ an embedding } [0, 1] \times \{1, 2, \dots, n\} \text{ w.r.t.} \\ \text{ + orientation consistency} \hookrightarrow 2\Sigma \\ \text{ s.t. each component of } \Sigma \text{ has nonempty } \partial \}$



These form a symmetric monoidal category: glue all
 outgoing open boundaries to incoming open boundaries.

$\mathcal{G} =$ dg symmetric monoidal category given by taking
 chains on $M_O(n, n)$. (rational coefficients)

Find variant: open-closed TFT
 $M_{OC}((n_0, n_c), (m_0, m_c))$
 $= \{ \text{a RS, } S' \times \{1, \dots, n_c + m_c\} \amalg [0, 1] \times \{1, \dots, n_0 + m_0\}$
 $\hookrightarrow \partial\Sigma$
 no connected component has all of its boundary
 closed & outgoing }

Def \mathcal{OC} : chains on M_{OC} : objects = $\mathbb{Z}_{\geq} \times \mathbb{Z}_{\geq 0}$

Technical point: if $F: \mathcal{C} \rightarrow \text{Car}_\infty \Rightarrow$
 maps $F(n) \otimes F(m) \rightarrow F(n+m)$

$\mathcal{O} \hookrightarrow \mathcal{OC} \hookleftarrow \mathcal{C}$
 require quasi-isomorphism

Theorem 1. There is a homotopy equivalence of categories
 between Open TFTs \mathcal{F} & Frobenius A_∞ algebras A_F
 2. If F is an open TFT
 there is a homotopy universal open-closed TFT
 $\mathbb{L}\mathbb{Z}_\infty F$; this gives an associated
 closed theory $j^* \mathbb{L}\mathbb{Z}_\infty F$.
 & $H_* (j^* \mathbb{L}\mathbb{Z}_\infty F(n)) = HH_*(A_F)^{\text{on}}$
 $[A_F = A_\infty \text{ algebra associated to } F]$

[Note: deformations compatible with Frobenius
 structure \leftrightarrow cyclic homology \rightsquigarrow
 can make Frobenius A_∞ algebra into
 dg Frobenius ...]

Corollary If A is A_∞ Frobenius \Rightarrow
 there are operators
 $H_*(m_c(\gamma_n)) \otimes HH_*(A)^{\text{on}} \rightarrow HH_*(A)^{\text{on}}$

1. M compact simply connected $\Rightarrow H^*(M)$
 has structure of A_∞ Frobenius algebra

using homological perturbation lemma \rightarrow transformation $f^*(\mu)$
 $HH_*(H^*(M)) = H^*(LM)$ (rational homotopy type + Poincaré duality)

$$H_*(M_{c(n, n)}) \otimes H^*(LM)^{\otimes n} \xrightarrow{f^*} H^*(LM)^{\otimes n}$$

--- this should be the string topology operators at least for M simply connected.

2. X smooth compact projective variety, $G(X) = 0$

$\Rightarrow D_\infty^b(X)$ ∞ -version of derived category of sheaves on X .
 $\text{Hom}(E, F) = \bigoplus \text{Ext}^i(E, F)$

Frobenius structure \leftrightarrow Serre duality

$$\text{Hom}(E, F) \otimes \text{Hom}(F, E) \rightarrow K$$

A ∞ structure

Should have $HH_n(D_\infty^b(X)) = \bigoplus_{i,j=0} H^{-i}(X, \mathcal{R}_X^j)$

\Rightarrow action of $H_*(M_{c(n, n)})$ on these.

[B-model mirror to part of GW theory]

3. Fukaya category.

X contact symplectic, F : $\text{Fuk}(X)$ a Calabi-Yau ∞ -category
 Get operators $H_*(M_{c(n, n)}) \otimes HH_*(F)^{\otimes n} \rightarrow HH_*(F)^{\otimes n}$

Using GW theory, can construct
 $H_*(M_{c(n, n)}) \otimes H_*(X)^{\otimes n} \rightarrow H_*(X)^{\otimes n}$

Theorem If a theory of open-closed GW invariant exists
 then there is a map $HH_*(F) \rightarrow H_{*+d}(X)$
 s.t. obvious diagram commutes.

[Kontsevich conjecture: $HH_*(F) \xrightarrow{\sim} H_*(X)$: diagonal Lagrangian
 Floer cohomology]

Technical point: A an A_{∞} algebra, \langle , \rangle on A of degree d , need to use charts with coordinates in a local system on $M(n,n)$

$$HC^*(C) = \mathbb{C}[[t]]$$

... deformations of C as Frobenius algebras are ∞ -dim: can make trivial but not compatibly with Frobenius structure.

Proof of Theorem - Geometric part: gen + relations
description of C \rightleftarrows find quasi-isomorphic model for O .

$$N(r,s) = \{ \Sigma \text{ Riemann surface with boundary, } r \text{ marked points on } \partial\Sigma \}$$

$$\textcircled{*} \rightarrow \textcircled{*} \Rightarrow \textcircled{\textcircled{*}} \text{ give 'marked' dots.}$$

$$\bar{N}(r,s) = \{ \Sigma \text{ as above with nodes on } \partial\Sigma \text{ distinct from marked pts} \}$$

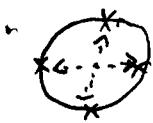
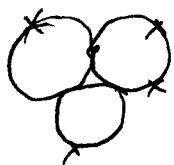
$$\textcircled{\textcircled{\Sigma}} \xrightarrow{\text{double}} \textcircled{\textcircled{\Sigma}} \text{ real version of node} \dots$$

$\bar{N}(r,s)$ related to real part of D-M moduli space \bar{m} .

... $\bar{N}(r,s)$ is an orbifold with corners (nodes like $IR_{\geq 0}^k$) when have k nodes

$\bar{m}(R)$ red pts of del. pt. \downarrow $C \times \bar{N}(r,s)$ form a category, g -is to O .

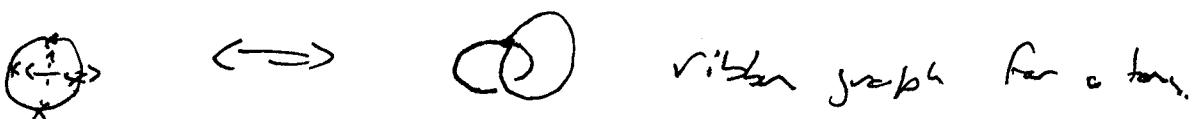
Theorem Let $D(r,s) \subseteq \bar{N}(r,s)$ be the set of $\Sigma \in \bar{N}(r,s)$ each of whose inner corners is a cusp. Then the inclusion $D(r,s) \hookrightarrow \bar{N}(r,s)$ is a weak homotopy equivalence.



Glue opposite ends together
get singular form.

$D(r,s)$ is a cell complex (stratification by homological type)

Each $\Sigma \in D(r,s) \rightsquigarrow$ a ribbon graph
with a vertex for each closed component
& edge for each node, tail for each marked vL



Why? Start with  its canonical
hyperbolic metric
with geodesic boundary, from density

\Rightarrow exponential map $\partial\Sigma \times R_{\geq 0} \rightarrow \Sigma$

Let $T \in R_{>0}$ be the first rank where $\text{Exp}(\partial\Sigma, T)$
is singular. Check only singularities are nodes

$$\sum \text{Exp}(\partial\Sigma \times [0, T]) \subset \partial \bar{N}$$

\rightsquigarrow deformation retraction of connected surfaces

$\bar{N}_{g,h}(r,s)$ onto its boundary.

-- breaks down precisely when we have discs



now decompose the boundary (normalize):

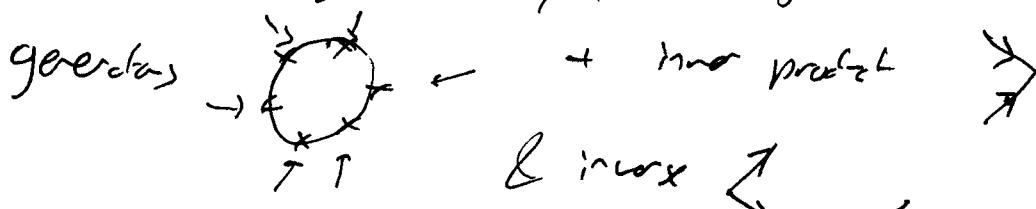


boundary is parametrized
by moduli of
normalization,
so apply metric f.
(we get discs
& cylinders, del with
exception)



So inductively get weak homotopy equivalence with
graphs which are discs \hookrightarrow ribbon graphs!
Dual to usual picture!

$D(r,s)$ topological category, with generators & relations.



$D(r) \cong$ Stasheff polytope K_{r+1}

$d(\text{fundamental chain of } \bullet) = \begin{cases} + \bullet \bullet \\ - \bullet \bullet \end{cases}$ splittings
... --> Axiom relation.

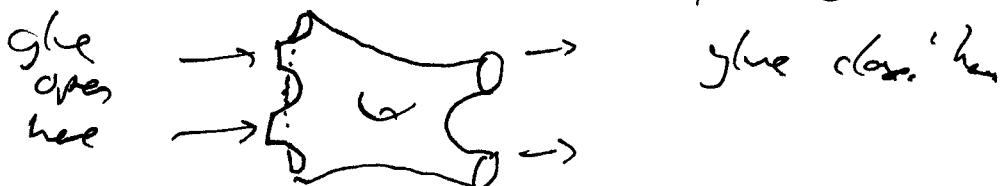
$$\mathcal{O} \xrightarrow{\quad 2 \quad} \mathcal{Q} \leftarrow \mathcal{C} \quad \begin{matrix} \text{calculate homology of} \\ j^* \amalg 2_* F \end{matrix}$$

• represent this composed functor by a bimod.

Consider $\mathcal{C} - \mathcal{O}$ bimodule

$$\mathcal{OC}((n_0, 0), (0, m_0))$$

only incoming open & only outgoing closed



$$j^* \amalg 2_* F = \mathcal{OC}((\bullet, 0), (0, \bullet)) \underset{\mathcal{O}}{\otimes} \overset{\mathbb{L}}{F}$$

explicitly: oriented

residue \mathcal{OC} by $\oplus \mathcal{OC} \otimes \mathcal{O}^\text{op}$

(bar resolution), flat resoltn.

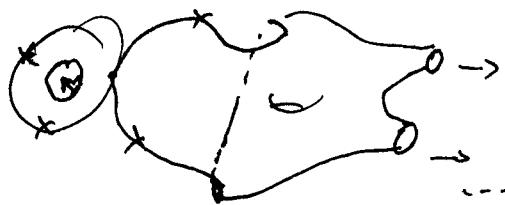
Need to describe $\mathcal{O}C$ as a right \mathcal{O} -module:
explicit disc description

$$\bar{N}^{OC}(r,s) = \left\{ \begin{array}{l} \text{with boundary, } r \text{ marked } A_1 \\ \text{S peripheral boundaries, distinct from marked } A_1 \end{array} \right\}$$

Notes only on boundary, S outgoing closed
outgoing closed boundaries are smooth

This moduli space is built up from discs & annuli:

$DA(r,s) \in \bar{N}(r,s)$: all mod components are discs
disjoint from outgoing closed boundaries, or
annuli, with precisely 1 boundary component closed, outgoing



.... \rightarrow generating & relating
description of $\mathcal{O}C$ as module over
disc category. : generators



relations from gluing.

Chain level: freely generated by



... gives Hochschild chain complex. $a_0 \otimes \dots \otimes a_n$

Differential: $\delta(\overset{\circ}{\bullet}) = \sum \pm \overset{\circ}{\bullet} \pm (\overset{\circ}{\bullet} \xrightarrow{\text{special map}} \overset{\circ}{\bullet})$

$$d(a_0 \otimes \dots \otimes a_n) = \sum \pm a_0 \otimes \dots \otimes a_m (g_{i+1}, \dots, g_{i+m}) \otimes \dots \otimes a_n$$

$$+ m_i(g_i \otimes a_{i+1} \dots \otimes a_r \otimes a_{i+1} \otimes \dots \otimes a_n)$$

$$\otimes a_{i+1} \otimes \dots \otimes a_{i+m}$$