D. Gaitsgory – Beilinson-Bernstein Localization 10/7/04

X smooth variety (char $=0$)

$\mathcal{D}_X$: sheaf of diff operators.

$\mathcal{L}$ line bundle $\Rightarrow$ sheaf $Diff(\mathcal{L},\mathcal{L}) =: \mathcal{D}'$
diffops on sections of $\mathcal{L}$ ... locally isomorphic to $\mathcal{D}_X$

Claim $Diff(\mathcal{L}^{\otimes n}, \mathcal{L}^{\otimes n})$ can be defined formally for any $n \in \mathbb{C}$!
More generally for any element of $Pic\, X \otimes_{\mathbb{Z}} \mathbb{C}$ get such a sheaf of twisted diffops ...

Construction Trivialize $\mathcal{L}$ over a Zariski cover U_i
$\mathcal{L}|_{U_i} \sim \mathcal{O}_{U_i} \Rightarrow \varphi_{ij} \in \mathcal{O}^*_{U_i \cap U_j}$ transitions

$Diff(\mathcal{L},\mathcal{L})|_{U_i} \sim \mathcal{D}_{U_i}$

On overlaps : φ_{ij} defines an automorphism of $\mathcal{D}_{U_i \cap U_j}$, ψ_{ij} as follows: ψ_{ij} acts as Id on functions
ψ_{ij} acts on vector fields:
$$\psi_{ij}(\xi) = \xi + \langle \xi, \frac{d\varphi_{ij}}{\varphi_{ij}} \rangle$$
$\frac{d\varphi_{ij}}{\varphi_{ij}} \rightsquigarrow d\log \varphi_{ij}$

.... ψ_{ij} comes from operator of conjugation on the noncommutative ring $\mathcal{D}$, conjugate by φ_{ij}.

$\rightarrow$ can define action of conjugation by a complex power of φ_{ij} : raising $\mathcal{L}$ to power n replaces $d\log\varphi_{ij}$ by $n\, d\log \varphi_{ij}$, so formula makes sense for $n \rightsquigarrow$ any $c \in \mathbb{C}$
– define patching data $\psi_{ij}(f) = f$
$$\psi_{ij}(\xi) = \xi + \langle \xi, c\, d\log \varphi_{ij} \rangle$$

— operator of conjugation by φ is a locally unipotent operator, so can raise to $\mathbb{C}$ power.

(Beilinson) $\mathcal{D}$ as bimodule over $\mathcal{O}$ living on diagonal $\Rightarrow$ can multiply by any function on formal nbhd of diagonal --- can take complex powers of function with value one on the diagonal in formal neighborhood: automorphism is multiplication by function $\varphi(x)/\varphi(y)$ on $\widehat{X \times X}_\Delta$.

(Dennis) Features of $\mathcal{D}'$: $\mathcal{D}$ is filtered by order of diffops & our automorphisms preserve the filtration $\Rightarrow$ $\mathcal{D}' = \bigcup_{i \geq 0} \mathcal{D}'_i$ filtered,

$\underline{\mathcal{D}'_0 = \mathcal{O}_X, \quad gr\, \mathcal{D}' \xrightarrow{\sim} gr\, \mathcal{D}} \xrightarrow{\sim} Sym_{\mathcal{O}_X}(T_X)$

G reductive $\supset B$ Borel $X = G/B$
Λ = lattice of cocharacters of $T \subset G$
$\Lambda^\vee$ = " " characters (= weights)
$\lambda \in \Lambda^\vee$, $\lambda \longmapsto \mathcal{L}^\lambda$ line bundle on $G/\underline{B}$ which is G-equivariant.

Define sheaves $\mathcal{D}^\lambda$ for arbitrary $\lambda \in \mathfrak{h}^*$

<u>Example</u> X variety with G action
M $\mathcal{D}_X$-module $\Longrightarrow$ $\Gamma(X, M)$ will be a module over $\mathfrak{g}$:
$G \circlearrowright X \Rightarrow \mathfrak{g} \longrightarrow Vect(X) \subset \Gamma(\mathcal{D}_X)$

Want to act by $\mathfrak{g}$ on sections of <u>twisted</u> $\mathcal{D}$-modules: what are natural conditions on a ring $\mathcal{D}'$ so that this will hold?

1. We want $\mathcal{D}'$ to be equivariant wrt G action on X,
ie $\mathcal{D}'$ G-equivariant as a quasicoherent sheaf
s.t. $\mathcal{D}' \otimes_{\mathcal{O}_X} \mathcal{D}' \longrightarrow \mathcal{D}'$ is G-equivariant

- This is not enough to get $\mathfrak{g}$ action on sections $\Rightarrow$

2. $\mathcal{D}'$ must be "of Harish-Chandra type" wrt G:
have homomorphism $\mathfrak{g} \xrightarrow{Lie} \Gamma(X, \mathcal{D}')$ s.t.
for d local section of $\mathcal{D}'$, $\xi \in \mathfrak{g}$ [Lie is G-equivariant]
$$act_\xi(d) = [Lie(\xi), d]$$ where act_ξ is
the "Lie derivative" coming from equivariance.

In this case, if M is a $\mathcal{D}'$-module, $\Gamma(X, M)$ is
a module over $\mathfrak{g}$ (for this don't use property 1.)

~~In practice~~:
More generally: [Harish-Chandra pair]:
1. A associative algebra with algebraic action of G
by automorphisms
2. $Lie : \mathfrak{g} \longrightarrow A$ homomorphism s.t. $\xi(a) = [Lie(\xi), a]$
+ ask Lie <u>G-equivariant</u>
- automatic if G connected

Example: $\mathcal{L}$ G-equivariant line bundle on X $\Rightarrow$
$Diff(\mathcal{L}, \mathcal{L})$ will be of Harish-Chandra type
... $\mathfrak{g}$ maps to $Diff_1(\mathcal{L}, \mathcal{L})$ by
differentiating sections, giving rise to <u>Lie</u> map.

Claim: In this case $\mathcal{D}' = Diff(\mathcal{L}^c, \mathcal{L}^c)$ is also
of Harish-Chandra type

$\Rightarrow$

$\Gamma : \mathcal{D}^\lambda\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}.$

Different POV on $\mathcal{D}^\lambda$ $\qquad X = G/B$

Consider $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_X$. What structures does this carry?

$T'_X := \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow T_X$

1. T'_X is a quasicoherent sheaf
2. T'_X acts on functions! $\quad T'_X \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow \mathcal{O}_X$

$\mathcal{O}_X$-linear $\qquad$ diff op of order 1 / vector field

3. $T'_X \otimes_{\mathbb{C}} T'_X \xrightarrow{[,]} T'_X$ $\quad$ diff op of order 1 wrt both variables.

A Lie algebroid on X is a quasicoh sheaf T^*_X with a homomorphism $\kappa : T'_X \to T_X$ & a bracket $[,]$ s.t.

$[f\xi_1, \xi_2] = f[\xi_1, \xi_2] - \xi_1 \cdot \xi_2(f)$

Lie algebroid : Lie groupoid :: Lie algebra : Lie group

look at kernel of κ, $\mathring{T}_X$: $\mathcal{O}_X$ - Lie algebra

(so e.g. fibers are Lie algebras)

$0 \to \mathring{T}_X \to \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_X \to T_X \to 0$ $\quad$ in flag variety case

is

$\mathfrak{b}_X$ = coherent sheaf on G/B with fiber at $x \in X$ is the Borel subalgebra $\mathfrak{b}_x$ corresponding to x = stabilizer.

Note : $\mathfrak{b} \subset \mathfrak{g}$ not an ideal, but $\mathfrak{b}_X \subset \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_X$ is an ideal (wrt not fiberwise bracket)

$\qquad\qquad\qquad\qquad \mathfrak{g}_X$

$0 \to \mathfrak{n}_X \to \mathfrak{b}_X \longrightarrow \mathfrak{h} \otimes \mathcal{O}_X \longrightarrow 0$

Fix $\lambda \in \mathfrak{h}^* \implies \mathfrak{h} \otimes \mathcal{O}_X \xrightarrow{\lambda} \mathcal{O}_X$ $\quad$ extend λ to $\mathfrak{b}_X$.

Define algebroid $T_X^\lambda = g_X \oplus_{b_X} \mathcal{O}_X$ coproduct

where b_X acts on $\mathcal{O}_X$ via $b_X \longrightarrow h \otimes \mathcal{O}_X \xrightarrow{\lambda} \mathcal{O}_X$
-- identify two images of b_X.

– e.g. if $\lambda = 0$ get $g_X / b_X \oplus \mathcal{O}_X$.

In general $0 \to \mathcal{O}_X \to T_X^\lambda \xrightarrow{\kappa} T_X \to 0$
(pushout of $0 \to b_X \to g_X \to T_X \to 0$ by λ).

Claim $\mathcal{D}^\lambda$ is "universal enveloping ~~algebra~~ $\mathcal{D}$-algebra" of T_X^λ:
in particular $\mathcal{D}_1^\lambda \simeq T_X^\lambda$ ....

— For any algebroid $\Rightarrow$ sheaf of associative algebras
$T_X \longmapsto \mathcal{D}_X$ ....

1. T' arbitrary Lie-algebroid $\Rightarrow U(T')$ generated by
$\mathcal{O}_X, T'$ with obvious relations & $\xi f - f\xi = \kappa(\xi)(f)$
$\xi_1 \xi_2 - \xi_2 \xi_1 = [\xi_1, \xi_2]$
— filtered, with $gr\, U(T') = Sym_{\mathcal{O}_X}(T')$

2. To obtain $\mathcal{D}^\lambda$ from $U(T^\lambda)$ need to quotient
it by identifying $\mathcal{O}_X \hookrightarrow T^\lambda$ with unit $\mathcal{O}_X \hookrightarrow U(T^\lambda)$.

<u>In general</u>: TDOs $\mathcal{D}' \longleftrightarrow$ Lie algebroids $0 \to \mathcal{O}_X \to T' \to T \to 0$
$\mathcal{D}' = U(T')/1-1$ s.t. $1 \in \mathcal{O}_X$ is central

<u>Def</u>: A TDO is a q-coherent sheaf $\mathcal{D}'$ with $\mathcal{O}_X \to \mathcal{D}'$ & algebra structure + filtration s.t. $gr\, \mathcal{D}' \xrightarrow{\sim} Sym\, T_X$ as Poisson algebras.

TDO = $H^1(X, \Omega^1 \xrightarrow{d} \Omega^2)$

~~$H^1(X, \Omega^1 \to \Omega^2)$~~

Note: not all TDOs algebraically locally trivial...

$\Gamma : \mathcal{D}^\lambda\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}$:

[Note $\mathcal{D}^\lambda$ is of Harish-Chandra type by this construction, via $\mathfrak{g} \longrightarrow \mathcal{D}^\lambda_1 = \mathfrak{g}_X \otimes_{\mathfrak{b}_X} \mathcal{O}_X$].

example: δ-functions: δ_x

$x \in X$, take $\mathcal{D}_\lambda \subset \mathcal{D}_\lambda \otimes_{\mathcal{O}_X} \mathbb{C}_x$ skyscraper at x

($\mathcal{D}_\lambda$-modules set-theoretically supported at a point $\simeq$ Vect)

. What are $\Gamma(G/B, \delta_x)$ as $\mathfrak{g}$-module

Claim $\Gamma(G/B, \delta_x) = M(\lambda - 2\rho)$ Verma module
$= ~~\mathrm{Ind}~~ \; U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_x)} \mathbb{C}_{\lambda - 2\rho}$

-2ρ: passage from left to right...

$\longrightarrow$ consider $\mathcal{D}_\lambda$ as built out of Vermas at each point $x \in G/B$.

Think of twisted $\mathcal{D}$-module on G/B as measure, we're integrating Verma modules against this measure to get a $\mathfrak{g}$-module.

$\mathbb{C}[\mathfrak{h}^*]^W \cong Z(\mathfrak{g}) \subseteq U(\mathfrak{g})$ Harish-Chandra homomorphism —

characterized by property:

for $\mu \in \mathfrak{h}^* \Rightarrow$ homomorphism $\chi_\mu : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$

$\chi_\mu(a)$ equals scalar by which a acts on the Verma module $M(\mu - \rho)$

$\chi_\mu(a)$ should depend only on W-orbit of μ, so need $\chi_\mu(a) = \chi_{w\mu}(a)$ i.e. a acts by same scalar on $M(w\mu - \rho)$ all $w \in W$...

Lemma The homomorphism $Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g})$ factors through

$$\begin{array}{ccc} Z(\mathfrak{g}) & \hookrightarrow & U(\mathfrak{g}) \\ {\scriptstyle \chi_{\lambda-\rho}}\downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & \Gamma(G/B, \mathcal{D}^\lambda) \end{array}$$

Proposition $U(\mathfrak{g}) \longrightarrow \Gamma(G/B, \mathcal{D}^\lambda)$

$$U(\mathfrak{g}) \searrow U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathbb{C}_{\lambda-\rho} \xrightarrow{\sim} \Gamma(G/B, \mathcal{D}^\lambda)$$

Proof (of lemma): all fibers of $\mathcal{D}^\lambda$ are Verma modules, $\therefore$ get same scalar out of action of center ... determines element in each fiber coming from central diffops.

Independent of Borel: given $x_1, x_2 \in G/B$ compare $M_{x_1}(\mu)$ to $M_{x_2}(\mu)$: pick $g \in G$ s.t. $gx_1 = x_2$. g defines an automorphism of $\mathfrak{g}$, transforming $M_{x_1}(\mu)$ to $M_{x_2}(\mu)$ but center is invariant (since G is connected)

Proof (of proposition): Use associated graded argument [unavailable in odd dim] Prove by induction on filtrations that

$$(U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathbb{C})_i \xrightarrow{\sim} \Gamma(G/B, \mathcal{D}^\lambda_i)$$

Use: ... $\boxed{\text{Theorem}}$ moment map

$$\begin{array}{c} T^*G/B \\ \downarrow \pi \\ N(\mathfrak{g}^*) \subset \mathfrak{g}^* \end{array}$$

($\eta \in \mathbb{C}[\mathfrak{g}^*] \mapsto$ vector field η as function on T^*G/B) $\eta \in \mathfrak{g}$

$$N(\mathfrak{g}^*) = \mathrm{Spec}\left(\mathrm{Sym}\,\mathfrak{g} \otimes_{(\mathrm{Sym}\,\mathfrak{g})^G} \mathbb{C}\right)$$

induces an isomorphism

$$\pi^* : \mathcal{O}_{N(\mathfrak{g}^*)} \xrightarrow{\sim} \Gamma(T^*G/B, \mathcal{O})$$

... since π is proper: fibers are subvarieties of G/B. Equivalent to normality of nilpotent cone ... which is a complete intersection.

Different perspective: $T^*G/B = \{ x \in X, \xi \in \mathfrak{n}_x \} = \tilde{\mathcal{N}}$
Useful to consider $\tilde{\mathfrak{g}} = \{ x \in X, \xi \in \mathfrak{b}_x \}$

$$\begin{array}{ccc} \tilde{\mathcal{N}} & \longrightarrow & \tilde{\mathfrak{g}} \\ \downarrow & & \downarrow \\ \mathcal{N} & \longrightarrow & \mathfrak{g} \end{array}$$

generically W-Galois cover

$$\begin{array}{ccccccccc} 0 & \to & \Gamma(G/B, \mathcal{D}^\lambda_{i-1}) & \to & \Gamma(G/B, \mathcal{D}^\lambda_i) & \to & \Gamma(G/B, \mathrm{Sym}^i T) & \to & 0 \\ & & \wr\uparrow & & \uparrow & & \wr\uparrow & & \\ 0 & \to & U(\mathfrak{g})^\lambda_{i-1} & \to & U(\mathfrak{g})^\lambda_i & \to & \mathrm{gr}^i(U(\mathfrak{g})^\lambda) & \to & 0 \end{array}$$

Using $\mathrm{gr}\, U(\mathfrak{g})^\lambda = \mathcal{O}_{\mathcal{N}(\mathfrak{g}^*)}$

$\Gamma : \mathcal{D}^\lambda\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}_{\chi_{\lambda-\rho}}$

Suppose this is an equivalence. $\delta_x \longrightarrow M(\lambda - 2\rho)$
δ-fn is irreducible $\mathcal{D}$-module $\Rightarrow$ RHS better be irreducible!

Q: When is $M(\mu)$ irreducible?
A: $\mu + \rho$ must be antidominant

BGG criterion: $\langle \mu+\rho, \check{\alpha} \rangle \notin \mathbb{N}^+$ all positive coroots
[for integral μ this is equivalent to usual antidominance, negativity on simple roots]

incorrect statement - see next lecture

Theorem (Beilinson-Bernstein)
a. $\lambda - \rho$ is antidominant ~~iff~~ $\Rightarrow \Gamma$ is exact
b. λ is antidominant ~~iff~~ $\Rightarrow \Gamma$ is exact & faithful

Corollary Assume λ antidominant. Then $\Gamma : \mathcal{D}^\lambda\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}_{\lambda-\rho}$ is an equivalence

Proof: $\Gamma(X, M) = \mathrm{Hom}_{\mathcal{D}\text{-mod}}(\mathcal{D}^\lambda, M) \Rightarrow \mathcal{D}^\lambda$ is projective generator of our category $\Rightarrow \mathcal{D}^\lambda\text{-mod} \simeq \mathrm{End}(\mathcal{D}^\lambda)^\circ\text{-mod} = \Gamma(G/B, \mathcal{D}^\lambda)\text{mod}$

<u>D. Gaitsgory</u>: Beilinson–Bernstein II

$\lambda : \mathfrak{h} \to \mathbb{C} \implies \mathcal{D}_\lambda$ TDO on $G/B = X$;
G-equivariant, & have $\mathfrak{g} \to \Gamma(X, \mathcal{D}_\lambda)$
$\Rightarrow \quad \Gamma : \mathcal{D}_\lambda\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}$ (left $\mathcal{D}$-mods)

$x \in X \implies \mathcal{D}_\lambda \underset{\mathcal{O}_x}{\otimes} k_x$ left $\mathcal{D}_\lambda$-module
$\Gamma(X, (\mathcal{D}_\lambda)_x) = M(\lambda - 2\rho)$ Verma module

$\lambda \mapsto \chi : Z(\mathfrak{g}) \to \mathbb{C}$ $\quad U(\mathfrak{g})_\chi := U(\mathfrak{g}) \underset{Z(\mathfrak{g})}{\otimes} \mathbb{C}_\chi$
Proved $U(\mathfrak{g})_\chi \xrightarrow{\sim} \Gamma(X, \mathcal{D}_\lambda)$
– relied on classical (ie assoc. graded) result:

$\mathcal{N} \times X \supset \tilde{\mathcal{N}} = T^*X$ proper map
$$\tilde{\mathcal{N}} \downarrow \mathcal{N} \quad \begin{cases} \bullet \ R^i\pi_* \mathcal{O}_{\tilde{\mathcal{N}}} = 0 & i > 0 \\ \bullet \ R^0\pi_* \mathcal{O}_{\tilde{\mathcal{N}}} = \mathcal{O}_{\mathcal{N}} \end{cases}$$

<u>Proof</u> $X \times \mathfrak{g} \supset \tilde{\mathfrak{g}} = \{x \in X, \xi \in \mathfrak{b}_x\}$ $\quad$ closed subvariety of $X \times \mathfrak{g}$
$$\tilde{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g}$$

Claim: $R^i \pi_* \mathcal{O}_{\tilde{\mathfrak{g}}} = 0, \ i > 0$
$$\pi_* \mathcal{O}_{\tilde{\mathfrak{g}}} = \mathcal{O}_{\mathfrak{g}} \underset{\mathcal{O}_{\mathfrak{h}/W}}{\otimes} \mathcal{O}_{\mathfrak{h}}$$
$$\mathfrak{g} \to \mathfrak{h}/W \leftarrow \mathfrak{h}$$

$\boxed{\text{This implies } \tilde{\mathcal{N}} \text{ claim, by base change:}}$
Cartesian diagram
$$\begin{tikzcd} & \tilde{\mathcal{N}} \arrow[dl] \arrow[dr] & \\ \tilde{\mathfrak{g}} \arrow[dr, "f"', "\text{flat}"] & & 0 \arrow[dl] \\ & \mathfrak{h} & \end{tikzcd}$$

<u>NOT true</u> that $\tilde{\mathfrak{g}} \underset{\mathfrak{h}/W}{\times} 0 = \tilde{\mathcal{N}}$:
LHS has nilpotents, RHS doesn't.

... rather we shall take preimage of nilpotent scheme inverse image of 0 in $\mathfrak{h}/W$ in $\mathfrak{h}$, & its preimage in $\tilde{\mathfrak{g}}$ is not $\tilde{\mathcal{N}}$ but length W thickening of it ...

$\tilde{\mathfrak{g}}$ is tautological $\mathfrak{b}$ bundle over X
has quotient $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$ bundle over X
but this bundle is trivial ---

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "\text{flat}"] \arrow[dr] & \mathfrak{h} \times X \arrow[r, "\text{proj}^n"'] \arrow[d] & \mathfrak{h} \\ & X & \end{tikzcd}$$

hence $\tilde{\mathfrak{g}} \xrightarrow{f} \mathfrak{h}$ is flat.

Consider $Rf_* \mathcal{O}_{\tilde{\mathfrak{g}}}$ - have no higher cohomology because have none for π, rest is affine map.

So we get: $Rf_* \mathcal{O}_{\tilde{\mathfrak{g}}} \overset{L}{\underset{\mathcal{O}_\mathfrak{h}}{\otimes}} \mathbb{C} = R\Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}})$ by flat base change

$$= \left(\mathcal{O}_\mathfrak{g} \underset{\mathcal{O}_{\mathfrak{h}/W}}{\otimes} \mathcal{O}_\mathfrak{h}\right) \overset{L}{\underset{\mathcal{O}_\mathfrak{h}}{\otimes}} \mathbb{C}$$

$$= \mathcal{O}_\mathfrak{g} \underset{\mathcal{O}_{\mathfrak{h}/W}}{\otimes} \mathbb{C} = \mathcal{O}_\mathcal{N}$$

Proof of $R^\bullet\pi_* \mathcal{O}_{\tilde{\mathfrak{g}}}$ calculation:

Enough to show (by Nakayama)

$R\pi_* \mathcal{O}_{\tilde{\mathfrak{g}}} \overset{L}{\underset{\mathcal{O}_\mathfrak{g}}{\otimes}} \mathbb{C} \simeq \mathcal{O}_\mathfrak{h} \underset{\mathcal{O}_{\mathfrak{h}/W}}{\otimes} \mathbb{C}$: isom on level of (derived) fibers suffices

(have map of complexes in one direction) + use $\mathbb{C}^*$ homogeneity by Nakayama

Problem! $\tilde{\mathfrak{g}} \to \mathfrak{g}$ not flat, can't apply base change

So use diagram

$$\begin{tikzcd} X \times \mathfrak{g} \arrow[d] & \tilde{\mathfrak{g}} \arrow[l, hook'] \\ \mathfrak{g} & \end{tikzcd}$$

Obtain $R\pi_* \mathcal{O}_{\tilde{\mathfrak{g}}} \overset{L}{\underset{\mathcal{O}_{\mathfrak{g}}}{\otimes}} \mathbb{C} = \bigoplus_i H^\bullet(X, \Omega^i[i])$

these are Tors in homology

By Hodge theory this is just $H^\cdot(X, \mathbb{C})$ which is $\mathcal{O}_{\mathfrak{h}} \underset{\mathcal{O}_{\mathfrak{h}/W}}{\otimes} \mathbb{C}$: (have map of algebras, with h actions, ...)

[Grauert-Riemenschneider: canonical bundle
Ginzburg: has no higher direct images for proper maps, & here on T^*G/B $\mathcal{O}$ is the canonical bundle, hence get $R^i\pi_* \mathcal{O} = 0$ ...]

[Use of $\tilde{\mathfrak{g}}$: have Koszul complex, so can calculate derived tensor .....]

[Affine case: $T^* Gr \longrightarrow \mathfrak{g}((t))$, no higher direct images, 0^{th} direct image is flat
However this classical statement doesn't give the quantum thing we want.]

Theorem

(we'll prove only $\Rightarrow$ direction)

a. If $\lambda-\rho$ is anti-dominant $\Leftrightarrow$ Γ is exact

b. If $\lambda-\rho$ is antidominant & regular $\Leftrightarrow$ Γ an equivalence

[μ is regular if $\langle\mu,\alpha\rangle \neq 0$ $\forall \alpha\in\Delta^+$
or $\Leftrightarrow$ $w(\mu)\neq\mu$ $\forall w\in W$]

[μ antidominant $\Leftrightarrow$ $M(\mu-\rho)$ irreducible]

c. If $\lambda-\rho$ is regular $\Leftrightarrow$ $R\Gamma$ is an equivalence of derived categories (if not regular have two Bruhat cells giving rise to same Verma module $\Rightarrow$ can't possibly have derived equivalence).

$\mathcal{D}_G = U(\mathfrak{g}) \underset{\mathbb{C}}{\otimes} \mathcal{O}_G$ (not commuting: $\mathfrak{g}$ differentiates $\mathcal{O}_G$)

... Heisenberg double of Hopf algebra

$l, r : U(\mathfrak{g}) \longrightarrow \mathcal{D}_G$ left & right invariant operators

how to write right action in terms of left action?

$\mathcal{D}_G = \mathcal{O}_G \underset{\mathbb{C}}{\otimes} U(\mathfrak{g})$

Write comultiplication

$$\begin{array}{ccc} U(\mathfrak{g}) \ni \xi & \xrightarrow{\Delta} & U(\mathfrak{g}) \underset{\mathbb{C}}{\otimes} \mathcal{O}_G \\ \uparrow & & \\ U(\mathfrak{g}) \ni \xi & \nearrow & \end{array}$$

adjoint

~~adjoint action~~ of Hopf algebra on its dual

-- to identify left & right invariant vector fields as a homomorphism of algebras (not anti-) need minus sign.

<u>Claim</u> $Z(l(U\mathfrak{g})) = r(U(\mathfrak{g}))$ left & right are each other's centralizers

~~$\mathcal{U}(\mathfrak{g})_l \otimes \mathcal{O}_G \cong \mathcal{D}_G$~~

[G acts on $U(\mathfrak{g})$ (algebraic rep) $\Rightarrow$ get $U(\mathfrak{g}) \to U(\mathfrak{g}) \underset{\mathbb{C}}{\otimes} \mathcal{O}_G$]

<u>Corollary</u>

$$\begin{array}{ccc} l(U(\mathfrak{g})) & \cap & r(U(\mathfrak{g})) \\ \parallel & & \parallel \\ l(Z(\mathfrak{g})) & & r(Z(\mathfrak{g})) \end{array}$$

center = bi-invariant diff ops same from left & right

— but actions of $Z(\mathfrak{g})$ from left & right are <u>not</u>

the same, but differ by action of involution of $Z(\mathfrak{g})$ given by $x \mapsto -x$ on $\mathfrak{h}$.

<u>Note</u> If M is a $\mathcal{D}_G$-module $\Rightarrow \Gamma(G, M)$ is a bimodule over $\mathfrak{g}$

<u>Pullback of $\mathcal{D}$-modules</u> $X_1 \xrightarrow{\pi} X_2$ smooth varieties

$\pi^*: \mathcal{D}_{X_2}\text{-mod} \longrightarrow \mathcal{D}_{X_1}\text{-mod}, \quad M \mapsto \mathcal{O}_{X_2} \underset{\pi^\cdot \mathcal{O}_{X_1}}{\otimes} \pi^\cdot M$

If $\mathcal{L}$ is a line bundle on X_2

$\mathcal{D}'_{X_2} = \mathcal{D}iff(\mathcal{L}, \mathcal{L}), \quad \mathcal{D}'_{X} = \mathcal{D}iff(\pi^*\mathcal{L}, \pi^*\mathcal{L})$

$\Rightarrow$ pullback functor $\pi^*: \mathcal{D}'_{X_2}\text{-mod} \longrightarrow \mathcal{D}'_{X_1}\text{-mod}$.

Take $G \xrightarrow{\pi} G/B$

For λ integral, $\pi^* \mathcal{O}^\lambda$ is trivial.

In fact for any λ have $\pi^*: \mathcal{D}_\lambda\text{-mod} \longrightarrow \mathcal{D}_G\text{-mod}$ ...

<u>On equivariant $\mathcal{D}$-modules</u>

Suppose $H \circlearrowright Y$ smooth variety, M $\mathcal{D}_Y$-module

When is M <u>equivariant</u>?

1. M must be endowed with H-equivariant $\mathcal{O}$-module structure, so that action map respects H structure!

$$\mathcal{D}_Y \underset{\mathcal{O}_Y}{\otimes} M \longrightarrow M \quad H\text{-equivariant}$$
(weakly equivariant)

Next for $\xi \in \mathfrak{g}$, $m \in \Gamma(Y, M)$ get two actions of ξ on m:

--- Lie derivative $\mathrm{act}_\xi(m)$ from differentiating equivariance

--- ξ gives vector field on Y, hence acts on M.

$\Rightarrow$ "defect of equivariance"
$\mathfrak{h} \underset{\mathbb{C}}{\otimes} M \longrightarrow M, \quad \xi, m \mapsto act_\xi m - \xi \cdot m$

<u>Claim</u> $\mathfrak{h}$ acts this way on M by $\mathcal{D}_Y$-module automorphisms (Lie algebra action by $\mathcal{D}_Y$ automorphisms)
~~Proof~~

2. M is called <u>strongly equivariant</u> if defect $= 0$.

<u>Example</u> a. $M = \mathcal{D}_Y$ itself is weakly equivariant by transport of structure, but defect map is right multiplication by ξ (as left $\mathcal{D}$-mod automorphisms).

b. $M = \mathcal{O}_Y$ <u>is</u> strongly equivariant

Suppose $Y//H = Z$ exists (in strongest possible sense)
$Y \xrightarrow[\pi]{} Z = Y//H$.
<u>Claim</u> (1) If M is a $\mathcal{D}_Z$-module $\Rightarrow \pi^* M$ is strongly equivariant.
:.. weak equivariance clear...
to check defect: look at case $Y = H \longrightarrow pt$, reduces to case b. above.

(2) $\pi^* : \mathcal{D}_Z\text{-mod} \longrightarrow$ strongly equivariant $\mathcal{D}_Y$-mod is an equivalence

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Let $H = B$, $Y = G$, $Z = G/B$ $\qquad G \xrightarrow{\pi} G/B$
& $\lambda : \mathfrak{h} \longrightarrow \mathbb{C}$ character
- can speak of B-equivariant $\mathcal{D}$-modules with defect λ:

Remark: a) can speak in general of H-equivariant $\mathcal{D}_Y$-modules with defect a given character $\lambda : \mathfrak{h} \to \mathbb{C}$ (above we had defect factoring $\mathfrak{b} \to \mathfrak{h} \to \mathbb{C}$

b.) λ produces a TDO on Z, $\mathcal{D}_Z^\lambda$, so that pullback gives an equivalence

$$\mathcal{D}_Z^\lambda\text{-mod} \xrightarrow{\sim} \text{weakly equivariant } \mathcal{D}_Y\text{-mod with defect } \lambda$$

Summarizing: $\pi^* : \mathcal{D}^\lambda\text{-mod on } G/B \to \mathcal{D}_G$-modules which are weakly B-equivariant with defect $\lambda : \mathfrak{b} \to \mathfrak{h} \to \mathbb{C}$.

So we calculate $\Gamma(G/B, M) = \Gamma(G, \pi^*(M))^B$

$$\| $$

$$\operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}, \Gamma(G, \pi^* M))$$

(since B connected, can go to just $\mathfrak{b}$-invariants)
– so far haven't used $\mathcal{D}$-module structure

But $\operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}, \Gamma(G, \pi^* M)) \xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}^{-\lambda}, \Gamma(G, \pi^* M))$

where $\mathfrak{b}$ acts on right on $\pi^* M$ via vector fields not by equivariance! on LHS action comes from $\mathcal{O}$-module B-equivariance, on RHS action comes from $\mathcal{D}_G$-module structure.

But this latter $\mathfrak{b}$ structure extends to $\mathfrak{g}$:

$$\Gamma(G/B, M) = \operatorname{Hom}_{\mathfrak{g}}(M(-\lambda), \Gamma(G, \pi^* M))^{*}$$

.... in fact on RHS both modules lie in category $\mathcal{O}$.

• Category $\mathcal{O} \subseteq \mathfrak{g}$-mod, corresponds to reps of $\mathfrak{g}$ s.t. the action of $\mathfrak{n}$ integrates to an algebraic action of N ($\Longleftrightarrow$ $\mathfrak{n}$ locally nilpotent)

& (Version 1) $Z(\mathfrak{g})$ acts semisimply

or

(Version 2) Fix an $\mathfrak{h} \subset \mathfrak{b}$, ask $\underline{\mathfrak{h}\text{ acts semisimply}}$

.. these categories are equivalent, but not tautologically ($\underline{\text{not}}$ as subcategories of $\mathfrak{g}$-mod

--- today: use version 2

e.g. $M(-\lambda) \in \mathcal{O}$ (in either sense)

$\underline{\text{Claim}}$ $\Gamma(G, \pi^*(M)) \circlearrowleft_{\mathfrak{g}} \in \mathcal{O}$ for any M

right action

... clearly integrable to N; defect trivial on N

action of $\mathfrak{h}$ differs from integrable $\mathfrak{h} \subset \mathfrak{b} = \mathrm{Lie}\, B$ action by the scalar λ, so still semi-simple.

So $\Gamma(G/B, M) = \mathrm{Hom}_G(M(-\lambda), \Gamma(G, \pi^* M))$

$\underline{\text{Lemma}}$ $M(\mu)$ is projective in $\mathcal{O}$ if $\mu + \rho$ is dominant

(dominant: Verma projective : nonnegative condition on roots,
antidominant: Verma irreducible : nonpos. condition on roots)

So ~~[illegible]~~ $\Gamma(G/B, -)$ is a composition of three exact functors π^*, $\mathrm{Hom}_G(-)$ & $\Gamma(G, -)$ (G affine)

$\Longrightarrow$ so get exactness of $\Gamma(G/B, -)$

Proof of Lemma Suppose $P \twoheadrightarrow M(\mu)$.
$Z(\mathfrak{g})$ acts locally finitely: can assume it acts on P with the same generalized character as on $M(\mu)$.
$\mathfrak{h}$ acts semisimply $\Rightarrow$ lift highest weight vector of $M(\mu)$ to P

$$\begin{array}{ccc} P & \longrightarrow & M(\mu) \\ \cup & & \cup \\ v'_\mu & \longmapsto & v_\mu \end{array}$$

Splitting $\Longleftrightarrow$ $\mathfrak{n}^+ \cdot v'_\mu = 0$ highest weight.

Why is this the case? $\mathfrak{n}$ acts loc nilpotently $\Rightarrow$ can find $X_{\alpha_1} \cdots X_{\alpha_n} \cdot v'_\mu \neq 0$ but highest weight vector (killed by $\mathfrak{n}_+$).
Let $\mu' = \mu + \alpha_1 + \cdots + \alpha_n$, weight of this high weight vector.

P has same infl character as $M(\mu)$ & $M(\mu') \longrightarrow P$
$v_{\mu'} \longmapsto$ h.w vector
so $w(\mu' + \rho) = \mu + \rho$ for some $w \in W$.

But dominance condition $\Longleftrightarrow$
$(\eta + \Sigma\Delta^+) \cap W(\eta) = \eta$: no such w except identity, so get splitting $\blacksquare$

(ref: intro of Frenkel–Gaitsgory)

Now how do we show in regular case that no objects are killed? need calculation.

1st method (BB) $H^\bullet(\mathfrak{n}^+, \Gamma(G/B, M)) \circlearrowleft \mathfrak{h}$

$$\cong \bigoplus_w H_{DR}(X_w, M|_{X_w}) \otimes \mathbb{C}^{w(\lambda+\rho)-\rho}$$

... so show that $\Gamma(G/B, M) \neq 0$:
$M \neq 0$ so can always find Schubert cell where fiber of M nonzero, $H_{DR} \neq 0$.

2nd method: Γ fully faithful in the derived category, using convolution functors — reduces to Exts between Vermas.

D. Gaitsgory: BB localization III

4/8/04

Y f.d smooth variety

$\mathcal{D}_Y$-mod: left $\mathcal{D}$-modules

$\ni \mathcal{M} \longmapsto \mathcal{M} \otimes \Omega^{top}$ right $\mathcal{D}$-module

$"\mathcal{M}^l" \qquad = \mathcal{M}^r$

Y_i smooth schemes of finite type

Suppose $\mathcal{Y} = \varprojlim Y_i$ projective limit of smooth schemes

& $Y_j \xrightarrow[p_{ij}]{} Y_i$ smooth. $p_i : \mathcal{Y} \to Y_i$

assume everything affine, $\mathcal{O}_\mathcal{Y} = \varinjlim \mathcal{O}_{Y_i}$, for now

$T_\mathcal{Y} = "\varprojlim" \mathcal{O}_\mathcal{Y} \underset{\mathcal{O}_{Y_i}}{\otimes} T_{Y_i}$: don't pass to limit object, which isn't q-coherent, but work with projective systems

Tangent spaces dangerous, cotangent better...

Example: V_i affine spaces, $V_j \xrightarrow{p_{ij}} V_i$

$V = \varprojlim_i V_i$

$\mathcal{O}_V = \mathrm{Sym}(V^*) \qquad V^* = \varinjlim V_i^* = \bigcup V_i^*$

V is pro, V^* is ind (usual) vector space - can think of V^* as primary object, $V = \mathrm{Spec}\ \underline{\mathrm{Sym}\ V^*}$ ---> polynomial algebra

V is product of countably many copies

$\Omega^\bullet_\mathcal{Y} = \varinjlim_i \mathcal{O}_\mathcal{Y} \underset{\mathcal{O}_{Y_i}}{\otimes} \Omega^\bullet_{Y_i}$

<u>Left $\mathcal{D}$-module</u>: $\mathcal{F}$ q-coherent sheaf, $\mathcal{F} \xrightarrow{d} \mathcal{F} \underset{\mathcal{O}_\mathcal{Y}}{\otimes} \Omega^1_\mathcal{Y}$

satisfying Leibniz & $d^2 = 0$

--- module with integrable connection.

For <u>any</u> scheme (not nec smooth, fin type) have notion of connection, integrable connection on an $\mathcal{O}$-module.

-- notion becomes reasonable under some kind of smoothness

Rewrite as action of vector fields:

$T_Y \otimes_{\mathbb{C}} \mathcal{F} \to \mathcal{F}$, taken in sense of projective systems. Won't factor through some particular i, but will do so for any given section of $\mathcal{F}$.

Example Let $\mathcal{F}_i$ be a left $\mathcal{D}$-module on Y_i & $\mathcal{F} = \mathcal{O}_Y \otimes_{\mathcal{O}_{Y_i}} \mathcal{F}_i$ $= p_i^* \mathcal{F}_i$ will be $\mathcal{D}_Y$-module.

... Naive pullback is left $\mathcal{D}$-module pullback.

Claim: any left $\mathcal{D}_Y$-module can be represented as $\varinjlim_i p_i^* \mathcal{F}_i$ $\mathcal{F}_i$ $\mathcal{D}_{Y_i}$-modules with embeddings $p_{ij}^*(\mathcal{F}_i) \hookrightarrow \mathcal{F}_j$

--- since $d(f)$ involves only finitely many elements of Ω^1_Y for $f \in \mathcal{F}$, all come from some fixed Y_i.

Let T_Y^{top} = topological inverse limit $= \varprojlim T_{Y_i}$
topological vector space.

$\Rightarrow$ action is a continuous map $T_Y^{top} \otimes \mathcal{F} \to \mathcal{F}$

Typical element of T_Y^{top}: $\sum_k f_k \cdot X_k$ $f_k \in \mathcal{O}_Y$

$X_k \in T_Y^{top}$ collection of vector fields tending to 0.

$m \in \mathcal{F} \Rightarrow$ $\sum_k f_k X_k \cdot m$ becomes finite sum by continuity

e.g. $V^* = \mathrm{Span}\{x_1, x_2, \dots\}$ $V = \mathrm{Spec}\ \mathbb{C}[x_1, x_2, \dots]$

$V = \overline{\mathrm{Span}} \{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots\}$ completed span of the $\frac{\partial}{\partial x_i}$

Any vector field is an infinite sum $\sum_k f_k \frac{\partial}{\partial x_k}$

$\frac{\partial}{\partial x_k} \longrightarrow 0$ as $k \to \infty$ by definition, so

Can act any such $\sum_k f_k \frac{\partial}{\partial x_k}$ on any polynomial — latter involves only fin. many x_i

$\mathcal{D}$-module here $\longleftrightarrow$ module over Weyl algebra generated by $v \in V$, $v^* \in V^*$

$[v, v^*](m) = \langle v, v^* \rangle \cdot m$.

$\left\{ \begin{array}{l} V^* \otimes F \to F, \\ V^{top} \otimes F \to F \text{ continuous} \end{array} \right.$

Does not make sense to act on the right by such a expression ...

[Consider $\mathrm{Sym}(V^*) \otimes V^{top}$: noncompleted tensor product] acts on any given left $\mathcal{D}$-module

$m \cdot \sum f_k \frac{\partial}{\partial x_k}$ doesn't make sense: f_k's do not tend to zero. Lose no topology

∴ use Leibniz: $m \cdot \sum f_k \frac{\partial}{\partial x_k}$

$$\sum \Big(\frac{\partial}{\partial x_k} f_k + \underbrace{\frac{\partial f_k}{\partial x_k}}_{\substack{\infty \text{ sum of functions} \\ \text{answer makes no sense!}}} \Big)$$

<u>Ind-schemes</u>

$Y = \text{"}\varinjlim\text{"}\, Y_i$ $\qquad$ $Y_i \hookrightarrow Y_j$ closed embeddings of smooth schemes of finite type

example $V = \text{"}\varinjlim\text{"}\, V_i$

⋮

functions on such an ind scheme is a topological algebra $\longleftrightarrow$ projective family $\mathcal{O}_Y = \text{"}\varprojlim\text{"}\, \mathcal{O}_{Y_i}$

$\mathcal{O}_V = \overline{\mathrm{Sym}}\, V^*$ completed.

What is a quasicoherent scheme?

· two kinds of $\mathcal{O}$-modules: $*$, $!$ type

$*$ $\mathcal{O}_Y$-module: $\mathcal{M}_i$ on Y_i: q-coh sheaves,

& isomorphisms $K_{ij}^*(\mathcal{M}_j) \simeq \mathcal{M}_i$. NOT abelian category!

Examples: · $\mathcal{O}_Y$ · $\{\mathcal{M}_i = \varinjlim T_{Y_j} \otimes_{\mathcal{O}_{Y_j}} \mathcal{O}_{Y_i}\}$

T_Y // tangent sheaf

$T_Y|_{Y_i}$ is honest q-coh sheaf

$!$-modules: in the affine situation, these are discrete modules over $\mathcal{O}_Y$: $\mathcal{O}_Y^{top} \otimes_{\mathbb{C}} \mathcal{M} \to \mathcal{M}$

$\mathcal{M}_1^* \otimes_{\mathcal{O}} \mathcal{M}_2^!$ is a $!$-module

$!$-modules: union of things where action factors through some $\mathcal{O}_{Y_i}$.

A right $\mathcal{D}$ module on Y is a $!$-sheaf $\mathcal{F}$ together with $\mathcal{F} \otimes_{\mathcal{O}_Y} T_Y \to \mathcal{F}$ s.t. usual axioms of right action hold.

$V = \mathrm{Span}\{\frac{\partial}{\partial x_i}\}$ $\quad V^* = \overline{\mathrm{Span}}\{x_i\}$

$\mathcal{D}_Y$-module $\leftrightarrow$ module $\mathcal{F}$ for Weyl algebra $W(V,V^*)$:

$V \otimes_{\mathbb{C}} \mathcal{F} \to \mathcal{F}$ $\quad V^{*top} \otimes_{\mathbb{C}} \mathcal{F} \xrightarrow[\text{continuous}]{} \mathcal{F}$

& $[v, v^*] = -\langle v, v^* \rangle$:

$m \cdot \sum f_k \frac{\partial}{\partial x_k}$ makes sense since $f_k \to 0$, so $m f_k = 0$ $k \gg 0$

but $\sum f_k \frac{\partial}{\partial x_k} \cdot m$ doesn't make sense

& Leibniz rule introduces infinite anomaly.

Mixed setting $\mathbb{V} = \varprojlim_i \varinjlim_j V_{ij}$

action only of linear functions & constant vector fields!

Definition A $\mathcal{D}$-module on $\mathbb{V}$ is a vector space $\mathcal{F}$ on which have $\mathbb{V}^{top} \otimes_{\mathbb{C}} \mathcal{F} \to \mathcal{F}$, $\mathbb{V}^{*top} \otimes \mathcal{F} \to \mathcal{F}$

continuous actions (put either $[V, V^*] = \langle V, V^* \rangle^{\mathbb{C}}$ or opposite) categories are equiv.

Here $\mathbb{V}$ is a Tate vector space.

e.g. $\mathbb{V} = \mathbb{C}((t))$ $\mathbb{V}^* \simeq \mathbb{C}((t))$ topological dual

Will be able to act only by <u>some</u> vector fields ...

... will only act on discrete objects here.

Tate vector space: topological vector space of a certain type.

e.g. • V discrete vector space $V = \varinjlim V_i$ V_i fin dim

• V profinite dim $V = \varprojlim V_j$

V is of Tate type if it can be represented as a direct sum $V_1 \oplus V_2$ V_1 discrete, V_2 profinite

e.g. $\mathbb{C}((t)) = \mathbb{C}[[t]] \oplus t^{-1}\mathbb{C}[t^{-1}]$

A lattice $\underset{\text{closed}}{L \subseteq \mathbb{V}}$: L profinite $\mathbb{V}/L$ discrete

(compact subspace)

colattice: closed discrete, co compact.

$\mathbb{V}^*$ topological dual is still Tate.

Goal: define $\mathcal{D}$-modules on $G((t))$

For any scheme Z, $Z[[t]]$ is a scheme $\varprojlim Z[t]/t^n$

$\mathrm{Hom}(\mathrm{Spec}\, A, Z[[t]]) = \mathrm{Hom}(\mathrm{Spec}\, A[[t]], Z)$

$A((t))$ exists as an ind-scheme if Z is affine.

~~G~~$G((t))$ group ind-scheme

$\mathcal{O}_{G((t))}$: algebra of functions: topological algebra
$\mathfrak{g}((t))$: Tate Lie algebra

Two actions $\mathfrak{g}((t)) \hat{\otimes} \mathcal{O}_{G((t))} \longrightarrow \mathcal{O}_{G((t))}$ $\quad Lie_{l(x)}$ left vector fields, $Lie_{r(x)}$ right

& adjoint action of $G((t))$ on $\mathfrak{g}((t))$:

$$\mathfrak{g}((t)) \longrightarrow \mathfrak{g}((t)) \hat{\otimes} \mathcal{O}_{G((t))}$$

.. can recover adjoint action from this.
Can probably do everything for reasonable group ind-schemes (with Lie algebra a Tate vector space)
- Morally have notion of indscheme modelled on Tate vector space. - two reasonable such notions

Change notation: G Tate group indscheme, $\mathfrak{g}$ Tate Lie algebra

$$\mathfrak{g} \hat{\otimes} \mathcal{O}_G \simeq TG \simeq \mathfrak{g} \hat{\otimes} \mathcal{O}_G$$
$$x \longmapsto l(x) \qquad r(x) \longleftarrow x$$

Same for 1-forms $\mathfrak{g}^* \hat{\otimes} \mathcal{O}_G \simeq \Omega^1_G \simeq \mathfrak{g}^* \hat{\otimes} \mathcal{O}_G$

Def A $\mathcal{D}$-module on G is a vector space F (discrete)

$\mathcal{O}_G \otimes F \longrightarrow F$ continuous
$\mathfrak{g} \otimes F \xrightarrow{a_l} F$ (left vector fields)
impose commutation between them:
$[a_l(x), f] = Lie_{l(x)} f$ ("left module")

[M smooth fin dim have $\mathcal{D}_M\text{-mod} \xrightarrow{\text{forget}} \mathcal{O}_M\text{-mod}$, $\otimes \omega^{top}$: $\mathcal{D}_M\text{-mod} \to \mathcal{D}_M\text{-mod}^{right} \to \mathcal{O}_M\text{-mod}$
doesn't 2-commute for ω^{top} nontrivial

algebraic tensor $\mathfrak{g} \otimes \mathcal{O}_G$ will act on $\mathcal{F}$ but answer doesn't extend to the completion TG

$TG = \mathfrak{g} \hat{\otimes} \mathcal{O}_G$ doesn't act ... even in case $G =$ a Tate vectorspace $\mathbb{V}$

ie action $\mathfrak{g} \otimes \mathcal{O}_G \otimes \mathcal{F} \to \mathcal{F}$ not continuous!

Want however to act, on any $\mathcal{F}$, by ~~right invariant~~ vector fields coming from right action

... will produce Tate central extension $0 \to \mathbb{C} \to \mathfrak{g}_{can} \to \mathfrak{g} \to 0$ such that:

Statement $\exists$ canonical action a_r on $\mathcal{F}$ of $\mathfrak{g}$-can s.t. ("right action")

- $[a_r(x), f] = \mathrm{Lie}_{r(x)} f$
- $[a_\ell(x), a_r(x)] = 0 \quad \forall x, y \in \mathfrak{g}$

Finite dimensions: have notion of $\mathcal{D}_G \supset$ right invariant, $\cup$ left invariant vector fields

$\Rightarrow U\mathfrak{g} \hookrightarrow \mathcal{D}_G \hookleftarrow U\mathfrak{g}$ s.t. each is other's centralizer

An analog of this holds in infinite dimensions:
have universal algebra acting on all $\mathcal{F}$'s,
& ~~the~~ $U\mathfrak{g}$, $U\mathfrak{g}_{can}$ each other's centralizer
(or some completions / quotients).

Have Lie algebra $0 \to \mathcal{O}_G \to \mathcal{D}_G^{\leq 1} \to TG$
maybe not surjective

$\mathcal{D}_G$: algebra generated by $\mathcal{O}_G$ & left vector fields

Have

D. Gaitsgory: BB localization V 11/15/04

Y variety ΠT_Y odd tangent bundle

$\cdot \mathcal{D}_{\Pi T_Y} \supset \mathcal{O}_Y, \Pi T^*_Y, \Pi T_Y, T_Y$

$f \quad \iota^*(x^*) \quad \iota(x) \quad Lie_x$

with relations 1. $[Lie_x, ?] = Lie_x(?)$

2. $[\iota(x), \iota^*(x^*)] = \langle x, x^* \rangle$

3. $Lie_{f \cdot x} = f\, Lie_x - df \cdot \iota(x)$

example: $Y = G$, $\Pi T_G = G \times \Pi \mathfrak{g}$

$i_r(x) \quad x \in \mathfrak{g}$

$\mathcal{D}_{\Pi T_G} = \mathcal{D}_G \hat{\otimes} Cliff(\mathfrak{g}, \mathfrak{g}^*)$

$\mathfrak{g} \overset{l}{\underset{r}{\rightrightarrows}} TG \supset \mathfrak{g}_r$, $\mathfrak{g}_l \subset TG$ left & right invariant vector fields

Construct $\mathfrak{g}_{can} \xrightarrow{a_r} \mathcal{D}_G$

$\mathfrak{g}_{can}$ central extension defined in terms of $\mathfrak{g}_{can} \xrightarrow{Cl} Cliff(\mathfrak{g}, \mathfrak{g}^*)$

$\Rightarrow a_r(x) = Lie_{r(x)} - Cl(x) \in \mathcal{D}_G$

sends $\mathfrak{g}_{can} \longrightarrow \mathcal{D}_G$

---

deRham differential $d \in \mathcal{D}_{\Pi T_Y}$, $d^2 = 0$

1. $[d, f] = df$

2. $[d, \iota_x] = Lie_x$ Cartan formula

<u>Lemma</u> Y reasonable ind-scheme $\exists!\, d \in \mathcal{D}_{\Pi T_Y}$ satisfying these relations

Proof Let X_i be a topological basis for T_Y as an $\mathcal{O}_Y$-module (and assuming $dx_i^* = 0$)

x_i^* -- dual basis in Ω_Y

$d = \sum_i \left(i^*(x_i^*) \cdot \mathrm{Lie}_{x_i} + i^*(dx_i^*)\, \iota(x_i) \right)$

where dx_i^* are 2-forms, coming from differential of x^* : claim we already know how to do this

$= \sum_i \left(\mathrm{Lie}_{x_i}\, i^*(x_i^* + \iota(x_i)\, \iota^*(dx_i^*) \right)$ ∎

If $\mathcal{F}^{DR}$ is any module over $\mathcal{D}_{\Pi T_Y}$, it acquires a differential.

Fin dim case: if $\mathcal{F}^{DR} = \mathcal{F}^{\ell} \otimes \Omega^{\bullet}_Y$ (left $\mathcal{D}$-module)

or $\mathcal{F}^{DR} = \mathcal{F}^{r} \otimes \Lambda(T_Y)$ (right $\mathcal{D}$-module)

$\Rightarrow$ d goes over to the standard deRham differential.

<u>Digression</u> If M is a module over $\mathfrak{g}_{-\mathrm{can}}$ - canonical extension $\Rightarrow$

$M \otimes \mathrm{Spin}$ (spin rep of Clifford) carries a canonical differential, functorial in M:

i.e $\exists !\ d \in U'(\mathfrak{g}_{\mathrm{can}}) \hat{\otimes} \mathrm{Cliff}(\mathfrak{g}, \mathfrak{g}^*)$

(U': set central element to 1) s.t.

0. $d^2 = 0$

1. $[d, \iota^*(x^*)] = \iota^*(\Delta(x^*))$ (inside Cliff) $\quad \Delta : \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*$

2. $[d, \iota(x)] = \mathrm{Lie}\, x$ where $\mathrm{Lie} : \mathfrak{g} \to U\mathfrak{g}_{-\mathrm{can}} \hat{\otimes} \mathrm{Cliff}$
(level $-$can) (level can)

... in the f.d. case have two standard models

for Spin: $\Lambda \mathfrak{g}^*$, $\Lambda \mathfrak{g}$ $\Rightarrow$ $M \otimes \mathrm{Spin}$ is $C^{\bullet}(M)$ the standard complex

$$M \otimes \Lambda \mathfrak{g}^* \overset{\text{cohomology}}{=} C^\bullet(\mathfrak{g}, M) \qquad M \otimes \Lambda \mathfrak{g} \overset{\text{homology}}{=} C_\bullet(\mathfrak{g}, M)$$

.. differ by $\Lambda^{top}\mathfrak{g}$. Intermediate case (wrt any subspace of $\mathfrak{g} \Rightarrow$ Lagrangian of $\mathfrak{g} \oplus \mathfrak{g}^*$) get semiinfinite cohomology.

Take group G setting

Start from $\mathcal{F}$, a $\mathcal{D}_G$-module,

$\mathcal{F}^{DR} = \mathcal{F} \otimes \mathrm{Spin}$, a $\mathcal{D}_{\Pi T_G}$-module

Observe: The deRham differential coming from the $\mathcal{D}_{\Pi T_G}$ point of view = the standard differential for $\mathcal{F}$ as $\mathfrak{g}_{\text{-can}}$-module

In the finite dim case:
compute $\mathcal{F}^{DR}$ .. take global sections of $\mathcal{F}$ as $\mathfrak{g}_r$ module, & calculate its standard complex:

$$\mathcal{F}^{DR} = C^\bullet(\mathfrak{g}, \mathcal{F})$$

--- this is why deRham cohomology of a semisimple group gives Lie algebra cohomology!

$$\mathcal{F} = \mathcal{O}_G : \quad H_{DR}(G, \mathcal{O}_G) = C^\bullet(\mathfrak{g}, \mathcal{O}_G)$$

$$\uparrow \text{ quasi-isomorphism for } G \text{ reductive}$$

$$C^\bullet(\mathfrak{g}, \mathbb{C})$$

--- all other pieces of $C^\bullet(\mathfrak{g}, \mathcal{O}_G)$ have wrong infinitesimal character to contribute.

Beilinson: anomalies are a rule of life.

Anomaly in fin dim case: G group:

G fin dim $\Rightarrow$ can make $\mathcal{O}_G$ a right $\mathcal{D}$-module, by setting ~~Lie$_{l(x)}\cdot 1 = 0$~~ $1\cdot l(x) = 0$ $x \in \mathfrak{g}$ & extending uniquely.

$\mathcal{O}_G$ is naturally weakly equivariant wrt $G\times G$

It is strongly equivariant wrt left copy of G (since $1\cdot l(x) = 0$)

For right action: the defect from strong equivariance is given by the modular character (how $\mathfrak{g}$ acts on $\Lambda^{top}\mathfrak{g}$)

This is a reflection of the anomaly in the finite dimensional world.

Modular character $G \to \mathbb{C}$ is a 1-cocycle $H^1(\mathfrak{g}, \mathbb{C})$

Loop anomaly: 2-cocycle $H^2(\mathfrak{g}, \mathbb{C})$

Back in ∞ dim: restore symmetry of left & right.

We defined $\mathcal{D}$-modules via left action of vector fields $\Rightarrow$ $\mathcal{D}_{G,l}$-mod.

Similarly can define via right action of vector fields $\mathcal{D}_{G,r}$-mod.

$\boxed{\text{Claim:}}$ $\mathcal{D}_{G,l}$-mod $\simeq$ $\mathcal{D}_{G,r}$-mod but not preserving the forgetful functor to vector spaces.

Why? To recover $\mathcal{F}$ from $\mathcal{F}^{DR}$, take

$$\mathcal{F}_l = \mathrm{Hom}_{Cliff(\mathfrak{g},\mathfrak{g}^*)}(\mathrm{Spin}, \mathcal{F}^{DR}) \quad \text{where}$$

Cliff acts via $i(r(x)), \iota^*(r(x^*))$

Alternatively could use left vector fields

$\mathcal{F}_r = \mathrm{Hom}_{Cliff}(\mathrm{Spin}, \mathcal{F}^{DR})$ using $i(l(a)), \varepsilon^*(l(a))$

To compare: use Clifford algebra over $\mathcal{O}_G$

$\mathrm{Cliff}_{\mathcal{O}_G}(T_G, T_G^*)$: $\mathrm{Hom}_{Cliff}(\mathrm{Span} \otimes_{\mathbb{C}} \mathcal{O}_G, \mathcal{F}^{DR}) = \mathcal{F}_l$

-- recall for $L \subseteq \mathfrak{g} \Rightarrow$ module

Spin_L induced from trivial rep of $\Lambda^{\cdot}(\mathfrak{g}/L) \otimes \Lambda(\mathfrak{g}^*/L^{\perp})$

$$\bigcap$$

$$\mathrm{Cliff}(\mathfrak{g}, \mathfrak{g}^*)$$

Can use an L which varies depending on the point in the group: conjugate by $g \in G$ the standard choice of L:

$\mathrm{Spin}_L \otimes_{\mathbb{C}} \mathcal{O}_G$ appears in the def of $\mathcal{F}_l$

but in def of $\mathcal{F}_r$ L varies over group, not constant...

-- in finite dimensions this doesn't matter, but in ∞ dims it does --

---

Suppose given extension $0 \to \mathbb{C} \to \mathfrak{g}' \to \mathfrak{g} \to 0$

Define $\mathcal{D}_G'$-modules: $(\mathfrak{g}', \mathcal{O}_G)$ modules

with the same axioms as before using left action of $\mathfrak{g}$. What will act on the right in this case?

Let $\mathfrak{g}'_{-can}$ = Baer sum $\mathfrak{g}_{-can} - \mathfrak{g}'$

<u>Theorem</u> On any $\mathcal{D}_G'$-module $\mathcal{F}$ have a canonical $\mathfrak{g}'_{-can}$ action: $\mathfrak{g}'_{-can} \xrightarrow{a_r} \mathcal{D}_G'$

$$\tilde{x} \longmapsto x$$

$$0 \to \mathcal{O}_G \to \mathcal{D}_G' \to T_G \to 0$$

Need $\mathcal{D}_G'$ to be an $\mathcal{O}_G$-bimodule:
$f\tilde{x} - \tilde{x}f = \mathrm{Lie}_x(f)$.

This doesn't define a continuous biModule structure on $\mathcal{O}_G \oplus T_G$. This is not an algebraic in ∞ dim.

For any ring $\mathcal{D}_Y$ of differential operators, get $\mathcal{D}_Y'$ with $0 \to \mathcal{O}_Y \to \mathcal{D}_Y' \to T_Y \to 0$

A)

1. $\mathcal{D}_Y'$ is an $\mathcal{O}_Y$-bimodule s.t.
   $f\tilde{x} - \tilde{x}f = \mathrm{Lie}_x(f)$ for $\tilde{x} \in \mathcal{D}_Y'$, $x \in T_Y$ its proj.
2. $\mathcal{D}_Y'$ is a Lie algebra & $f\tilde{x} - \tilde{x}f = [f, \tilde{x}]$.

From such can reconstruct full algebra $\mathcal{D}_Y$ as enveloping algebra $/ 1 - \mathbb{1}$.

B) Other type of creatures $0 \to \mathcal{O}_Y \to \tilde{T}_Y \to T_Y \to 0$ s.t.

1. $\tilde{T}_Y$ is a symmetric $\mathcal{O}_Y$-bimodule (ie $\underline{\mathcal{O}_Y\text{-module}}$).
2. $\tilde{T}_Y$ is a Lie algebra $[\tilde{x}, f] = \mathrm{Lie}_x(f)$

Finite dimensions: forgetting right $\mathcal{O}$-action gives an equivalence between type A & type B objects -- can define right action by formula A1.

In ∞ dim this fails due to continuity:
$\tilde{T}_Y = \mathcal{O}_Y \oplus T_Y$ as type B does not define an object of type A, won't get continuous right action.

A ~~B~~ is a torsor over ~~A~~ B, so don't expect a map either way in infinite dimensions. : ~~to~~ can add second types, or can add type B to A

... B's are Picard category, $A + B \Rightarrow A$.
(above we forgot some of the axioms of A which ensure there is no map $A \to B$).

From $0 \to \mathbb{C} \to \mathfrak{g}' \to \mathfrak{g} \to 0$ get B datum:

$$0 \to \mathbb{C} \otimes \mathcal{O}_G \to \mathfrak{g}' \hat{\otimes}_G \mathcal{O}_G \to \mathfrak{g} \hat{\otimes}_{\mathfrak{g}} \mathcal{O}_G \to 0$$

$$\mathbb{C} \otimes \mathcal{O}_G = \mathcal{O}_G, \qquad \mathfrak{g} \hat{\otimes} \mathcal{O}_G = T_G$$

Our $\mathcal{D}_G' = \mathcal{D}_G \underset{\text{Baer}}{+} \mathfrak{g}' \otimes_{\mathbb{C}} \mathcal{O}_G$
type (A) + type (B)

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_G \arrow[r] & \mathcal{D}_G'^{\,1} \arrow[r] & T_G \arrow[r] & 0 \\ & & \mathfrak{g}'_{can} \arrow[u, "r"] \arrow[r] & \mathfrak{g} \arrow[u, "r"'] & \end{tikzcd}$$

Must construct a map $-\mathfrak{g}' \to \mathfrak{g}' \otimes \mathcal{O}_G$ the simple type B object:
[Problem: TDOs don't have de Rham complex, can't use Clifford / de Rham structure as before -- so instead use Baer sum constructions.]

$\mathfrak{g}' \to \mathfrak{g}' \hat{\otimes} \mathcal{O}_G$ action, use it to define

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_G \arrow[r] & \mathfrak{g}' \otimes \mathcal{O} \arrow[r] & T \arrow[r] & 0 \\ & & -\mathfrak{g}' \arrow[u] & & \end{tikzcd}$$

hence by Baer sum get our theorem.. ∎

D. Gaitsgory : The convolution action 11/23/04

1. G f.d group $\circlearrowright X$ variety

$D(\mathcal{D}_G\text{-mod}) \times D(\mathcal{D}_X\text{-mod}) \longrightarrow D(\mathcal{D}_X\text{-mod})$
$\quad \overset{\in}{F} \qquad\qquad \overset{\in}{M}$

$G \times X \xrightarrow{act} X$ take $act_*(F \boxtimes M) \in D(\mathcal{D}_X\text{-mod})$
$= F * M$

e.g. $F = \delta_g$ ... category of all $\mathcal{D}$-modules, not necessarily
δ-functions holonomic, so have only $*$-pushforward
(& only !-pullback)

$\delta_g * M$: translation of M by $g: X \longrightarrow X$
... so action integrates together action of translations

2. $\mathfrak{g}$-mod : $D(\mathcal{D}_G\text{-mod}) \times D(\mathfrak{g}\text{-mod}) \longrightarrow D(\mathfrak{g}\text{-mod})$
$F, \; M \longmapsto F * M$

example $F = \delta_g$:
$\delta_g * M = g \cdot M$: G acts on $\mathfrak{g} \Rightarrow$ can twist
$\mathfrak{g}$-modules by action of g to get new module.

Remember: any function is a "linear combination of δ-functions" :
in some sense ... as integral $f = \int_X f(x) \cdot \delta_x$

... same in some sense for $\mathcal{D}$-modules.

3. Most general context :

A associative algebra, $G \circlearrowright A$ ($\Leftrightarrow A \longrightarrow \mathcal{O}_G \otimes A$ coaction)
+ Harish-Chandra data
$\phi: \mathfrak{g} \longrightarrow A$ s.t. 1. ϕ G-equivariant
2. $x \in \mathfrak{g}$, $[\phi(x), a] = Lie_x(a) (= x \cdot a)$

-- i.e. $\mathfrak{g}$ action is inner.

[$G \circlearrowright V \Leftrightarrow V \longrightarrow \mathcal{O}_G \otimes V$: action of $g \cdot v$ for
fixed v & varying g $\longleftrightarrow$ map $G \longrightarrow V$]

In H-C setting have
$$D(\mathcal{D}_G\text{-mod}) \times D(A\text{-mod}) \longrightarrow D(A\text{-mod})$$
$$\quad\ F \qquad\qquad\quad M$$

Examples: 1. $G \curvearrowright X$ affine, $A = \mathcal{D}_X$

2. $A = U(\mathfrak{g})$

<u>Construction</u>

$F \tilde{\boxtimes} M \Rightarrow$ a module over $\mathcal{D}_G \otimes A$

as a vector space just take $F \otimes M$, but with action

1. A acts via $A \longrightarrow \mathcal{O}_G \otimes A$ followed by natural action on $F \otimes M$

2. $\mathcal{O}_G$ acts as $f \otimes 1$

3. right vector fields $\tilde{a}_r$ $r(\mathfrak{g})$ acts diagonally : $x \mapsto a_r(x) \otimes 1 + 1 \otimes \phi(x)$
$x \in \mathfrak{g}$ action on F

<u>Exercise</u>: $\Rightarrow \mathcal{D}_G, A$ commute.

action of left vector fields:

$$\tilde{a_\ell}(x) = a_\ell(x) \otimes 1 - (\text{mult} \otimes \phi) \circ \Delta(x)$$

$\Delta : \mathfrak{g} \longrightarrow \mathcal{O}_G \otimes \mathfrak{g}$ adjoint action

$F * M = H_{DR}(G, F \tilde{\boxtimes} M)$ $\cancel{A}$. complex of A-modules

is

$$(F \tilde{\boxtimes} M)^{DR} = F \otimes M \otimes \Lambda(\mathfrak{g}^*) :$$

a complex of A-modules ... acquires differential commuting with A ... standard complex for $\mathfrak{g}$ acting on $F \otimes M$ by $\tilde{a}_r$

The differential d on $F \otimes M \otimes \Lambda(\mathfrak{g}^*) \simeq C^\cdot(\mathfrak{g}, F \tilde{\boxtimes} M)$
- standard <u>differential</u> for $\tilde{a}_r$ action, commutes with A.

Example: $F = \mathcal{O}_G$, $F \otimes M$ = M-valued functions on G Maps (G, M), has action

~~again $M = A \ni \alpha \mapsto (g \mapsto g\alpha g^{-1})$~~

∞-dim case: action is $A \longrightarrow \mathcal{O}_G \hat{\otimes} A$,

$\mathcal{O}_G$ is pro-algebra (G ind) F, M both discrete

A eg $= \widehat{U(\mathfrak{g})}$, everything else makes perfect sense.

$F * M = F \otimes M \otimes Spin$: make choice of Spin representation of Clifford algebra — $\infty/2$ complex.

Problem! in ∞ dim can't realize $\mathfrak{g}$-mod to $\mathcal{D}$-mod on G (weakly equivariant) - but in fin dim case we can.

M $\mathfrak{g}$-module, ϕ = action of $\mathfrak{g}$ on M [$\mathfrak{g}$ Tate Lie algebra]

$\hat{U}\mathfrak{g}$ = universal algebra acting on category of (discrete) Lie algebra

$\tilde{\varphi}$ = action of $\mathfrak{g}$ on $F \tilde{\boxtimes} M$

$F * M$ is given by the complex $F \otimes M \otimes Spin$

$\tilde{\phi} = (mult_{\mathcal{O}_G} \circ \phi) \circ \Delta$

Have another action of $\mathfrak{g}$, more naive:

recall F is $\mathfrak{g}$-bimodule - from $\mathcal{D}$-module L from H-C structure [left action]

$\longrightarrow$ on $F \otimes M$ still have left action of $\mathfrak{g}$, namely $a_\ell \otimes id$.

<u>Claim</u>: these two actions are in some sense homotopic:

$\exists$ action $\tilde{i}_\ell : \Lambda(\mathfrak{g}) \otimes C^\bullet(\mathfrak{g}, F \tilde{\boxtimes} M) \longrightarrow C^\bullet(\mathfrak{g}, F \tilde{\boxtimes} M)[1]$

Cartan contraction

s.t. $[d, \tilde{i}_\ell(x)] = \underbrace{\tilde{\phi}(x) - a_\ell(x) \otimes id}_{= \tilde{a}_\ell(x)}$ difference of the two actions of $\mathfrak{g}$.

fin dim situation : $H_\mathfrak{g}(F \otimes M)$ is a module, from left action - but this is <u>not</u> the $\tilde{\phi}$ action

bimodule, module

<u>Lemma</u>

Given two actions of $\mathfrak{g}$ which are homotopic in this sense (difference $= [d, -]$ where the homotopy operators anticommute) $\longrightarrow$ <u>quasi isomorphic</u> $\mathfrak{g}$-complexes.

... quism commuting with $\mathfrak{g}$-action

- Suppose $G \curvearrowright A$ but ϕ action is centrally extended!
$0 \to \mathbb{C} \to \mathfrak{g}' \to \mathfrak{g} \to 0$, $\quad \phi : \mathfrak{g}' \to A$

$\Rightarrow D(\mathcal{D}_G'\text{-mod}) \times D(A\text{-mod}) \to D(A\text{-mod})$

$F \tilde{\otimes} M$ is again a $\mathcal{D}_G \otimes A$-mod, so can still take its de Rham cohomology.

- Suppose $\mathfrak{g}', \mathfrak{g}''$ central extensions, $\mathfrak{g}' + \mathfrak{g}'' = \mathfrak{g}_{\text{-can}}$

M', M'' modules $\Rightarrow C^\cdot(\mathfrak{g}, M' \otimes M'')$ makes sense

<u>Lemma</u> Let F be a $\mathcal{D}_G'$-module

extension reflection principle
$$\left\{\begin{array}{l} C^\cdot(\mathfrak{g}, (F * M') \otimes M'') \\ \quad\quad \| \\ C^\cdot(\mathfrak{g}, M' \otimes (F * M'')) \end{array}\right.$$

$= \ldots$. $\langle M, N \rangle = C^\cdot(\mathfrak{g}, M \otimes N)$

$\langle M'', F * M' \rangle = \langle M'' * F, M' \rangle$ adjointness

<u>Pf</u>: both are $C^\cdot(\mathfrak{g}, M' \otimes F \otimes M'')$ using second action $a_\ell \otimes id$. - because both are same bimodule ... (ℓ 2 actions are homotopic). $\blacksquare$

Corollary/ <u>Example</u> $\mathfrak{g}' = \mathfrak{g}$, $\mathfrak{g}'' = \mathfrak{g}_{\text{-can}}$
$M' = \mathbb{C}$ $\quad M'' = M$

$\Rightarrow \quad C^\cdot(\mathfrak{g}, F * M) = C^\cdot(\mathfrak{g}, M) \otimes H_{DR}(G, F)$

since $F * \mathbb{C} = H_{DR}(G, F)$ with <u>trivial $\mathfrak{g}$</u> action

Example $F = \delta_g$: Get that $C^\cdot(\mathfrak{g}, -)$ is invariant under twisting ~~by~~ of modules by G action.

General story $\mathcal{C}$ abelian ~~variety~~ category with ∞ direct sums --> direct limits exist: eg Grothendieck category. G group (fin dim)

X affine variety $\mathcal{C}_X$ = objects of $\mathcal{C}$ over X = objects of $\mathcal{C}$ endowed with an $\mathcal{O}_X$-action

(could replace $\mathcal{O}_X$ by any associative algebra)

e.g. $\mathcal{C} = A$-mod $\mathcal{C}_X = A \otimes \mathcal{O}_X$-mod.

Weak action of G on $\mathcal{C}$ is a functor $\mathcal{C} \xrightarrow{act} \mathcal{C}_G$ + piece of data

[eg G finite: each element acts by a functor, & have associativity isomorphisms + cocycle condition]

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "act"] \arrow[d, "act"'] & \mathcal{C}_G \arrow[d, "act"] \arrow[dl, shift left] \arrow[dl, shift right] \\
\mathcal{C}_G \arrow[r] & (\mathcal{C}_G)_G = \mathcal{C}_{G\times G} \arrow[ul, "m^*"']
\end{tikzcd}$$

$m: G\times G \to G$

$Y \to X$ map $\Rightarrow$

$$\mathcal{C}_Y \underset{\text{forget}}{\overset{\otimes_{\mathcal{O}_X}\mathcal{O}_Y}{\rightleftarrows}} \mathcal{C}_X$$ right adjoint to forget

--- can tensor objects of $\mathcal{C}_X$ by $\mathcal{O}_X$-modules

datum is natural isomorphism

$act \circ act \simeq m^* \circ act$: associativity constraint on action

+ axiom: cocycle condition

Example $G \circlearrowright X$ any variety $\Rightarrow \mathcal{C} = \mathcal{O}_X$-mod carries weak action of G.

An ~~obj~~ equivariant object of $\mathcal{C}$ is an object $M \in \mathcal{C}$ with isomorphism $act^* M \simeq M \underset{\mathbb{C}}{\otimes} \mathcal{O}_G$ s.t. compatibility holds --- can prove this $act^* M$ is always flat/G

$\mathcal{C}^G :=$ category of equivariant objects --- categorical invariants

eg for G finite: all translations of M by group elements are same as M.

Can consider categories $\tilde{C}$ acted with an action of the tensor category Rep G

<u>Claim</u>: category C^G of equivariant objects is of this type:

M "$\otimes$" V is $M \otimes \underline{V}$ ($\underline{V}$ underlying vector space) as an object of C but $act^*(M "\otimes" V) \to M "\otimes" V \otimes \mathcal{O}_G$ is changed

$\Rightarrow$ 2 functor $\{G\text{-categories}\} \longrightarrow \{\text{Rep } G\text{-categories}\}$

<u>Claim</u> This 2-functor is an equivalence

ie for every $\tilde{C}$ acted on by Rep G can produce category Hecke$(G, \tilde{C})$, category acted on by G, & two procedures are inverse.

$Ob(\text{Hecke}(G, \tilde{C})) = M \in \tilde{C}$ + system of isomorphisms $M "\otimes" V \underset{\alpha}{\simeq} M \otimes \underline{V}$.

compatible with tensor product of V's.

G acts (naively) changing isomorphisms α by action on $\underline{V}$.

... acting by $\mathbb{C}[G]$ Hopf algebra or its dual, coaction vs action.
... duality for findim Hopf algebras.

Strong actions / Harish-Chandra action:
Let $G \circlearrowright C$ & suppose we're given the following additional data:

Let $G^{(1)} = \text{Spec}(\mathbb{C} \oplus \varepsilon \mathfrak{g}^*)/\varepsilon^2 = 0$
first infi nbhd of $1 \in G$

given $act^*(M)|_{G^{(1)}} \simeq M[\varepsilon]/\varepsilon^2$ infinitesimal trivialization of action

+ axiom for $G \circlearrowright G^{(1)}$ adjoint action

Example: Weak action $G \curvearrowright A$ automorphisms of algebra
Strong action of $\mathfrak{g} \to A$ H-C datum
($\mathcal{C}$ = A-modules)

Claim: In this context can define canonical action

Equivariant objects in old (weak) sense = weakly equivariant.

Strong equivariance: $\underbrace{act^*(M)|_{G^{(1)}} \underset{H\text{-}C}{\cong} M[\varepsilon]/\varepsilon^2}_{\text{equivariance}}$

ask two identifications to coincide.

For $\mathcal{C}$ with weak action have universal category with strong action
- just formally look at M + data $act^*|_{G^{(1)}} \cong M[\varepsilon]/\varepsilon^2$

Example: $\mathcal{C}$ = Vect, with trivial action of G
then this universal category is just Rep cat.