D. Gaitsgory : De Jong's Conjecture 10/23/04

$X/\mathbb{F}_q$ normal variety $\quad \rho : \pi_1(X) \to GL_n F \quad F = \mathbb{F}_\ell((t))$

$\bar\rho$ = restriction of ρ to $\overline{\pi_1}(X) \hookrightarrow \pi_1(X) \to \hat{\mathbb{Z}} \to 1$

Conj: Im $\bar\rho$ is finite

De Jong showed suffices to assume X is a complete curve, $\bar\rho$ is absolutely irreducible

Bun_n - moduli stack of GL_n bundles on X
De Jong shows his conjecture follows if to ρ can associate f_ρ F-valued function on $Bun_n \mathbb{F}_q$ which is a Hecke eigenfunction w/ eigenvalues determined by ρ.

Careful! coefficients not in characteristic zero!

$x \in |X|$ $\quad H^i_x \quad i = 1, \dots n$ Hecke operators on automorphic functions

$$\begin{tikzcd} & \mathcal{H}^i \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] \arrow[d, "s"] & \\ Bun_n & X & Bun_n \end{tikzcd}$$

$\mathcal{H}^i = (x, \mathcal{M}, \mathcal{M}', \mathcal{M} \overset{\beta}{\hookrightarrow} \mathcal{M}'$ s.t. $\mathcal{M}'/\mathcal{M}$ is scheme theoretically supported at x & is of length i)

$$H^i_x = \overrightarrow{h}_! \, \overleftarrow{h}^*(-) \cdot q^{\frac{i(n-i)}{2}}$$

Hecke eigenfunction: $H^i_x(f_\rho) = Tr(\Lambda^i(\rho(Frob_x))) \cdot f_\rho$

$\rho \longleftrightarrow E$ n-dim local system of F-vector spaces on X

Note! ℓ-adic sheaf is not sheaf in étale topology of $\mathbb{Q}_\ell$-v.spaces..

Similarly have notion of F-sheaves, similar systems of finite coefficient systems

Def $\mathcal{F} \in D(Bun_n)$ ($\mathcal{F}$-derived category) is called a *weak Hecke eigensheaf* wrt E if

$H^i(\mathcal{F}) \simeq \Lambda^i E \boxtimes \mathcal{F}$

[H=H'] **Digression** $\mathcal{F}$ is an eigensheaf wrt E if we are given an isomorphism
$H(\mathcal{F}) \xrightarrow{\sim} E \boxtimes \mathcal{F}$ s.t.
$(id_X \times H)\, H(\mathcal{F}) \xrightarrow{\sim} E \boxtimes E \boxtimes \mathcal{F}\,|_{X \times X \setminus \Delta \times Bun...}$
is compatible with the Σ_2-equivariant structure.

Introduce new stack

$$\begin{tikzcd} & Mod_n^d \arrow[dl, "h"'] \arrow[d, "s"] \arrow[dr, "?"] & \\ Bun_n & X^{(d)} & Bun_n \end{tikzcd}$$

$= (\mathcal{M}, \mathcal{M}', \beta : \mathcal{M} \hookrightarrow \mathcal{M}'$ s.t. $\mathcal{M}'/\mathcal{M}$ is of length d)

s takes data to $\det(\mathcal{M}'/\mathcal{M})$.

Locally picture is

$$\begin{array}{c} \mathfrak{gl}_d \\ \downarrow \\ \mathfrak{h}/W \end{array}$$

$E \rightsquigarrow$ Laumon's sheaf $\mathcal{L}_E^d$ on Mod_n^d :
$j : \mathring{X}^{(d)} \subset X^{(d)}$ complement of diagonals

(reg s-s elements)

$$\begin{tikzcd} \mathring{Mod}_n^d \arrow[r, hook, "j"] \arrow[d, "\mathring{s}"] & Mod_n^d \arrow[d] \\ \mathring{X}^{(d)} \arrow[r, hook, "j"] & X^{(d)} \end{tikzcd}$$

away from diagonal $\mathring{Mod}_n^d$ is just d lines in fibers
$\Rightarrow$ d projective spaces $\mathbb{P}^{n-1}$

So from E define by symmetric product $\mathring{E}^{(d)}$ on $\mathring{X}^{(d)}$

$j_{!*}\, \mathring{s}^*(\mathring{E}^{(d)}) =: \mathcal{L}_E^d$

Satake equivalence let $\mathcal{H}_x = (\mathcal{M}, \mathcal{M}', \beta : \mathcal{M} \simeq \mathcal{M}'|_{X \setminus x})$

$\Rightarrow$ F-representations $Rep(\check{GL}_n) \xrightarrow{\sim} Perv(\mathcal{H}_x)$
(Mirković-Vilonen)

$\mathcal{H}$ $\quad D, \mathcal{M}, \mathcal{M}' : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'|_{X \setminus |D|}$

$\downarrow$

$D \in X^{(d)}$ $\quad$ [note above stacks $\mathcal{H}'$ had $\mathcal{M} \subset \mathcal{M}'$: no poles, & we fixed degree of quotient $\leadsto$ components]

More generally:
$P^d_{\check{GL}_n}$ Perverse sheaves on $X^{(d)}$ with an action of $\check{GL}_n$

$$P^d_{\check{GL}_n} \longrightarrow Perv(\mathcal{H})$$

V : standard n-dim rep of $\check{GL}_n$ (over F)

$(V \otimes E)^{(d)} \in P^d_{\check{GL}_n} \longrightarrow Perv(\mathcal{H})$
goes to Laumon's sheaf
Note: char p must be careful defining symmetric powers! safest approach: middle extension from complement of diagonals.

Laumon's sheaf $\xrightarrow{\text{series of Fourier transforms}}$ $\mathcal{F}_E \in D(Bun_n)$
automorphic eigensheaf

modulo obstruction ; vanishing result.
Obstruction
For $1 \leq n' \leq n-1$ $\Rightarrow$ averaging functor
$$Av_E^d : D(Bun_{n'}) \longrightarrow D(Bun_{n'})$$

Need vanishing of Av_E^d identically for $d \gg 0$.

Change notations! $rk\, E = m > n$, work on Bun_n (which was n' before).

$$\mathrm{Av}_E^d = \overrightarrow{h}_!(\overleftarrow{h}^*(\mathcal{F}) \otimes \mathcal{L}_E^d)$$

<u>Theorem</u> $\mathrm{Av}_E^d \equiv 0$ if • $\mathrm{rk}\, E > n$

• $d > (2g-2)n\, \mathrm{rk}\, E$ • E is irreducible

"real theorem behind this":

<u>Theorem</u> Under same assumptions, Av_E^d is <u>exact</u> in sense of perverse t-structure.