1) Gaitsgory - Geometric Eisenstein Series 12/1/99

X curve as usual (/$\mathbb{C}$ for simplicity). $K = \mathbb{C}(X)$, $\mathbb{A}$, $\mathcal{O}_{\mathbb{A}}$. G reductive, $\pi_1[G,G] = 0$

$U \subset B \subset G$, T , Λ = coweights of $T = \mathrm{Hom}(\mathbb{G}_m, T)$

$$\mathrm{Bun}_B = \{ (\mathcal{F}_G, \mathcal{F}_T, \{\kappa^{\lambda^\vee}\}_{\Lambda^\vee}) \quad \kappa^{\lambda^\vee} : \mathcal{L}^{\lambda^\vee}_{\mathcal{F}_T} \xhookrightarrow{\text{line subbundle}} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_G} \text{ satisfy Plücker} \}$$

$$\mathrm{Bun}_G \xleftarrow{p} \mathrm{Bun}_B \xrightarrow{q} \mathrm{Bun}_T$$

Plücker: $\mu^\vee, \lambda^\vee \in \Lambda^\vee$
$$\begin{array}{ccc} \mathcal{L}^{\lambda^\vee}_{\mathcal{F}_T} \otimes \mathcal{L}^{\mu^\vee}_{\mathcal{F}_T} & \longrightarrow & \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_G} \otimes \mathcal{V}^{\mu^\vee}_{\mathcal{F}_G} \\ \uparrow & & \uparrow \\ \mathcal{L}^{\lambda^\vee+\mu^\vee}_{\mathcal{F}_T} & \longrightarrow & \mathcal{V}^{\lambda^\vee+\mu^\vee}_{\mathcal{F}_G} \end{array}$$
commutative

(κ's determined by those for fund. weights)

$\overline{\mathrm{Bun}}_B = \{ (\mathcal{F}_G, \mathcal{F}_T, \{\kappa^{\lambda^\vee}\}) \quad \kappa^{\lambda^\vee}$ injective map of sheaves $\}$ ... precisely injective sheaf map for every geometric point.

$\Rightarrow \bar{p}, \bar{q}$ to $\mathcal{F}_G, \mathcal{F}_T$.

$\mathrm{Eis} : \mathrm{Sh}(\mathrm{Bun}_T) \longrightarrow \mathrm{Sh}(\mathrm{Bun}_G)$ Sh: derived category of $\left\langle \begin{array}{l} \mathcal{D}\text{-mod } (\mathbb{C}) \\ \ell\text{-adic sh } (\mathbb{F}_q) \end{array} \right.$

$\mathrm{Eis}(\mathcal{S}) = \bar{p}_!(\bar{q}^*(\mathcal{S}) \otimes IC_{\overline{\mathrm{Bun}}_B})[\dim \mathrm{Bun}_T]$ (IC as integral kernel ...... - makes this self-dual ...)

Note: $\overline{\mathrm{Bun}}_B$ not smooth so $\bar{q}^*$ doesn't commute with Verdier duality
$\bar{p}$ proper on each component, $! = *$ ..

Theorem 1. $\mathcal{S} \to$ ~~[illegible]~~ $\bar{q}^*(\mathcal{S}) \otimes IC_{\overline{\mathrm{Bun}}_B}$ commutes with Verdier duality
(locally $\bar{q}$ decomposes into product ....)
2. $j : \mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B$. $\mathcal{S}$ perverse $\Rightarrow \bar{q}^*(\mathcal{S}) \otimes IC_{\overline{\mathrm{Bun}}_B}$ is just $j_{!*}(q^*(\mathcal{S}))$ [shift]

Suppose $E_{T^\vee}$ is $T^\vee$-local system on $X \xRightarrow{\text{abelian CFT}} \mathrm{Aut}_{E_{T^\vee}}$ automorphic perverse sheaf on Bun_T - just 1-d local system (with shift)

$\mathcal{F}^\circ_T$ trivial T-bundle. $\mathcal{F}_T = \mathcal{F}^\circ_T(\sum_i \mu_i x_i)$ $\mu_i \in \Lambda^\vee$

So fiber $(\mathrm{Aut}_{E_{T^\vee}})_{\mathcal{F}_T} = \bigotimes_i (E^{-\mu_i}_{T^\vee})_{x_i}$ $E^{\mu_i}_{T^\vee}$ assoc line bundle.

$T^\vee \hookrightarrow G^\vee \Rightarrow$ induce local systems $E_{T^\vee} \longrightarrow E_{G^\vee}$.

Theorem 1 $\mathrm{Eis}(\mathrm{Aut}_{E_{T^\vee}})$ is a Hecke eigensheaf wrt $E_{G^\vee}$.
(would fail if we didn't compactify Bun_B ...)

Theorem 2 (Functional equation) Assume $E_{T^\vee}$ is regular (GL_n: 1-d local systems are pairwise non-isomorphic .. G: every root get 1-d local system, want all nontrivial $\Rightarrow$ classical Eis has no poles)

$$\mathrm{Eis}(\mathrm{Aut}_{E_{T^\vee}}) \simeq \mathrm{Eis}(\mathrm{Aut}_{wE_{T^\vee}}) \quad \forall w \in W$$

Note: $E_{T^\vee} \not\simeq wE_{T^\vee}$ but induced G-bundles are.

Note: this isom is not canonical: need not just $w \in W$ but lift $\tilde{w} \in N(T) \subset G$. Monodromy (ambiguity) is given by action on $Eis(Aut_{E_{T^\vee}})$ of dual Cartan.

let $Eis'(Aut_{E_{T^\vee}}) = p_!(q^*(S) \otimes IC_{Bun_B})[\dim Bun_T]$: no compactification.

$Bun_T = \bigcup_{\mu \in \Lambda} Bun_T^\mu$. $Aut_{E_{T^\vee}} = \bigoplus_\mu Aut^\mu_{E_{T^\vee}}$.

Introduce generating series in K-group $\sum_\mu Eis(Aut^\mu_{E_{T^\vee}}) t^\mu \in K(Sh)[[t^\Lambda]]$

<u>Theorem 3</u> $\sum_\mu Eis(Aut^\mu_{E_{T^\vee}}) t^\mu = \sum_\mu Eis'(Aut^\mu_{E_{T^\vee}}) t^\mu \cdot \prod_{\alpha \in \Delta^+} L(\alpha(E_{T^\vee}), t^\alpha)$

where $L(\alpha(E_{T^\vee}), t^\alpha) = \sum_{n \geq 0} t^{n\alpha} \cdot H^*(X^{(n)}, (\alpha(E_{T^\vee}))^{(n)})$
abelian L-series (α char of $T^\vee$).
– easy modulo knowledge of IC_{Bun_B} –
stalks of <u>IC have simple conjectural descriptions</u> which is equiv. to Thm 3.

$x \in X$, K_x, O_x. $Gr_G = G_{K_x}/G_{O_x} = \bigcup_{\lambda \in \Lambda^+} \overline{Gr_G^\lambda}$ closures of G_{O_x}-orbits
dom. coweights
Sph_G = G_{O_x}-equiv perv. sheaves on Gr_G
$\simeq Rep\, G^\vee$. $IC_{\overline{Gr_G^\lambda}} \longleftrightarrow V_\lambda$.

Sph_G acts on $Sh(Bun_G)$: $V \in Rep_{G^\vee}$, $H(V, \cdot) : Sh(Bun_G) \to Sh(Bun_G)$
Hecke functors.
$G = T$: $Gr_T \simeq \Lambda$ <u>as sets</u> ($= T_{K_x}/T_{O_x}$), $Sph_T = \Lambda$-graded vect. spaces.
$H_T(\lambda, \cdot)$ is the pullback functor for $Bun_T \xrightarrow{m_\lambda} Bun_T$,
$m_\lambda(\mathcal{F}_T) = \mathcal{F}_T(-\lambda \cdot x)$.

<u>Theorem 1'</u> $V \in Rep\, G^\vee$, $S \in Sh(Bun_T)$
$H_G(V, Eis(S)) = Eis(H_T(Res^G_{T^\vee}(V), S))$
$\qquad\qquad\qquad\qquad\qquad \shortparallel$
($\Rightarrow$ Theorem 1) $\qquad \bigoplus_\lambda Eis(m_\lambda^*(S)) \otimes V(\lambda)$ $\leftarrow$ λ-weight subspace

<u>Satake map</u> (classical). $Eis' : C_c^\infty(G_{\mathbb{A}}/N_{\mathbb{A}}B_K) \to C_c^\infty(G_{\mathbb{A}}/G_K)$
$\shortparallel$
$Ind^{G_{\mathbb{A}}}_{B_{\mathbb{A}}}(C_c^\infty(T_{\mathbb{A}}/T_K))$

Unramified: take $G_{O_{\mathbb{A}}}$ invariants

Lemma If π is a representation of T_{K_x}, then
$\left(\mathrm{Ind}_{B_{K_x}}^{G_{K_x}}(\pi)\right)^{G_{O_x}} = \pi^{T_{O_x}}$ compatibly with Hecke actions:

$\exists\ \mathrm{Sat}: \underset{h}{H(G_{K_x}, G_{O_x})} \longrightarrow H(T_{K_x}, T_{O_x}) \simeq \mathbb{C}[\Lambda]$, $h \cdot v = \mathrm{Sat}(h)\cdot v$.

Formula: $\mathrm{Sat}(h) = \sum_{\mu\in\Lambda} \mathrm{Sat}^\mu(h)\,\mu$ .. to describe $\mathrm{Sat}^\mu(h)$
look at U_{K_x}-orbits in G_{K_x}/G_{O_x}, $\Lambda \ni \mu \longmapsto S_\mu = U_{K_x}\cdot\tilde\mu \subset \mathrm{Gr}_G$.
$\mathrm{Sat}^\mu(h) = \int_{S_\mu} h(g)\,dg$ (discrete set...)

Geometrically $\mathrm{Gr}_G = \bigcup_{\mu\in\Lambda} S_\mu$ locally closed ind-subscheme

$\mathrm{Sat}: \mathrm{Sph}_G \longrightarrow \mathrm{Sph}_T$. $\mathrm{Sat}(S) = \bigoplus_{\mu\in\Lambda} \mathrm{Sat}^\mu(S)$
$\quad\ \ \mathrm{Rep}_{G^\vee} \xrightarrow{\mathrm{Res}} \mathrm{Rep}_{T^\vee}$ $\qquad \mathrm{Sat}^\mu(S) = H_c^0(S_\mu, S|_{S_\mu})[\langle\mu, 2\check\rho\rangle]$

Cohomology lives only in one degree –
but this is hard!

Hecke for G: $\overline{\mathcal{H}}_G^\lambda = \{(\mathcal{F}_G, \mathcal{F}_G', \beta: \mathcal{F}_G \simeq \mathcal{F}_G'|_{X\setminus x}) \,/\, \forall \check\lambda \in \check\Lambda^+,$
$V_{\mathcal{F}_G}^{\check\lambda}(-\langle\lambda,\check\lambda\rangle x) \hookrightarrow V_{\mathcal{F}_G'}^{\check\lambda} \hookrightarrow V_{\mathcal{F}_G}^{\check\lambda}(-\langle w_0\lambda, \check\lambda\rangle x)$

$$\begin{tikzcd} & \overline{\mathcal{H}}_G^\lambda \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \end{tikzcd}$$

- $\overleftarrow{h}$ fibration with fibers $(\overleftarrow{h})^{-1}(\mathcal{F}_G) \sim \overline{\mathrm{Gr}}_G^\lambda$, w_0 = long word
$(\overrightarrow{h})^{-1}(\mathcal{F}_G') \sim \overline{\mathrm{Gr}}_G^{-w_0\lambda}$.

$$H_G(V^\lambda, S) = \overleftarrow{h}_*\left(\overrightarrow{h}^*(S) \otimes IC(\overline{\mathcal{H}}_G^\lambda)\right)[\dim \mathrm{Bun}_G]$$
$$\text{IS}$$
$S \,\tilde\boxtimes\, IC_{\overline{\mathrm{Gr}}_G^{-w_0\lambda}}$ twisted external tensor product (wrt G_O)

•
$$\begin{tikzcd} \mathrm{Bun}_B \arrow[d] & Z \arrow[l, dashed, "\phi"'] \arrow[d] \arrow[r, "\overrightarrow{h}'"] \arrow[dr, phantom, "\square"] & \overline{\mathrm{Bun}}_B \arrow[d, "\bar p"] \\ \mathrm{Bun}_G & \overline{\mathcal{H}}_G^\lambda \arrow[l, "\overleftarrow{h}"'] \arrow[r, "\overrightarrow{h}"] & \mathrm{Bun}_G \end{tikzcd}$$

Prop $\exists\ \phi$ (proper map) which makes the left square commutative.

Proof $Z = \{(\mathcal{F}_G, \mathcal{F}_G', \mathcal{F}_T', \beta, \{\kappa^{\check\lambda}\})\}$

$\phi(\text{this}) = (\mathcal{F}_G, \mathcal{F}_T, \{\kappa^{\check\lambda}\})$.
new $\mathcal{F}_T = \mathcal{F}_{T'}(w_0(\lambda)\cdot x)$

$V_{\mathcal{F}_G}^{\check\lambda} \sim V_{\mathcal{F}_G'}^{\check\lambda}$ on $X \setminus x$

So have $\mathcal{V}^{\lambda^\vee}_{\mathcal{F}_G} \sim \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_G'}$, $\quad$ Claim: $\kappa^{\lambda^\vee}$ is well-defined & regular (no roots).

$$\begin{array}{ccc} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_G} & \sim & \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_G'} \\ \kappa^{\lambda^\vee}\uparrow & & \uparrow \kappa'^{\lambda^\vee} \\ \mathcal{L}^{\lambda^\vee}_{\mathcal{F}_T} & \sim & \mathcal{L}^{\lambda^\vee}_{\mathcal{F}_T'} \end{array}$$

Example GL_2, $\lambda = \binom{1}{0}$. M rank two bundle

$$\begin{array}{ccc} \left(\begin{array}{c} M \hookrightarrow M' \\ \uparrow \\ \mathcal{L}' \end{array}\right) & \to & (\mathcal{L}' \hookrightarrow M') \\ \downarrow & \square & \downarrow \\ (M \hookrightarrow M') & & \\ \swarrow \quad \searrow & & \\ (M) \qquad & & (M') \end{array}$$

$\phi\left(\begin{array}{c} M \hookrightarrow M' \\ \uparrow \\ \mathcal{L}' \end{array}\right) = (M, \mathcal{L}'(-x))$, $\quad \mathcal{L}'(-x) \hookrightarrow M$

$\Lambda \supset \Lambda^+$ dominant

$\check{\Lambda}^{pos}$ positive coweights: pos. linear comb of pos. roots

(for general G reductive don't have $\Lambda^+ \subset \Lambda^{pos}$)

$\nu \in \Lambda^{pos} \Rightarrow i_\nu : \overline{Bun}_B \hookrightarrow \overline{Bun}_B$.

$i_\nu(\mathcal{F}_G, \mathcal{F}_T, \{\kappa^{\lambda^\vee}\}) = (\mathcal{F}_G, \mathcal{F}_T', \{\kappa'^{\lambda^\vee}\})$

where $\mathcal{F}_T' = \mathcal{F}_T(-\nu \cdot x)$, $\quad \kappa'^{\lambda^\vee} : \mathcal{L}^{\lambda^\vee}_{\mathcal{F}_T'} = \mathcal{L}^{\lambda^\vee}_{\mathcal{F}_T}(-\langle \nu, \lambda^\vee\rangle \cdot x) \hookrightarrow \mathcal{L}^{\lambda^\vee}_{\mathcal{F}_T} \xrightarrow{\kappa} \mathcal{V}^{\lambda^\vee}_{\mathcal{F}_G}$

- just introduce zero artificially.

Example GL_2 $(M, \mathcal{L} \hookrightarrow M) \to (M, \mathcal{L}(-nx))$

Theorem 1'' $\quad \phi_!(IC_Z) = \bigoplus_{\nu \in \Lambda^{pos}} i_\nu(IC_{\overline{Bun}_B}) \otimes V^\lambda(w_0 \cdot \lambda + \nu)$

$\Downarrow$

Theorem 1'

$\Downarrow$

Theorem 1

Note: $\hat{h}^{\to}$ locally trivial, so IC_Z comes from tensor of IC's:

Lemma $\quad IC_Z = {}'p^*(IC_{\overline{Gr}^\lambda_G}) \otimes {}'h^{\to *}(IC_{\overline{Bun}_B})[\dim Bun_G]$

Content of Thm 1'': map is stratified semismall.

Proof $\quad \nu \in \Lambda^{pos}$ introduce (1) ${}_{\geq\nu}\overline{Bun}_B = \mathrm{Im}(i_\nu(\overline{Bun}_B))$, closed

(2) ${}_\nu\overline{Bun}_B = \left({}_{\geq\nu}\overline{Bun}_B \setminus \bigcup_{\nu' > \nu} {}_{\geq\nu'}\overline{Bun}_B\right) \xrightarrow{j_\nu} \overline{Bun}_B$

$\quad \uparrow$

(3) ${}_\nu Bun_B = \mathrm{Im}(i_\nu(Bun_B))$

$(\mathcal{F}_G, \mathcal{F}_T, \kappa^{\lambda^\vee})$: (1) $\kappa^{\lambda^\vee}$ has zero of order $\geq \langle \nu, \lambda^\vee \rangle$ $\forall \lambda^\vee \in \Lambda^\vee$.

(2) " " " " " = " "

(3) + no zeros away from x.

<u>Proposition 1</u> a. $j_\nu^* \phi_! (IC_Z)$ lives in cohom. degrees ≤ 0.

b. $j_\nu^* \phi_! (IC_Z)|_{\overline{_{\geq\nu}Bun_B} - {}_\nu Bun_B}$ " " " " < 0.

c. $j_\nu^* \phi_! (IC_Z)|_{{}_\nu Bun_B} = V^\lambda(w_0(\lambda)+\nu)$ constant sheaf (up to shift)

Prop $\Rightarrow$ Theorem by decomposition theorem: a. $\Rightarrow$ pure perverse sheaf (direct sum) - by duality get deg ≤ 0 & ≥ 0..

b, c say what are the summands.

<u>Proof of Prop</u> $\nu, \nu' \in \Lambda^{pos}$, $\lambda' \in \Lambda^+$.

Introduce $Z^{\nu,\nu',\lambda'} \hookrightarrow Z$

$$= \phi^{-1}({}_\nu Bun_B) \cap (\overrightarrow{'h})^{-1}({}_{\nu'}Bun_B) \cap ('p)^{-1}(\mathcal{H}_G^{\lambda'})$$

Let us denote by $K^{\nu,\nu',\lambda'} = \phi_! (IC_Z|_{Z^{\nu,\nu',\lambda'}})$

(reformulate) <u>Proposition 2</u> a. $K^{\nu,\nu',\lambda'}$ lives in coh. deg ≤ 0, & inequality is strict unless $\nu' = \lambda' = 0$.

b. $K^{\nu,0,\lambda'}|_{\overline{{}_\nu Bun_B} - {}_\nu Bun_B}$ lives in coh deg ≤ 0.

c. $K^{\nu,0,\lambda}|_{{}_\nu Bun_B} = V^\lambda(w_0\lambda+\nu) \otimes$ constant sheaf

<u>Now use geometry of affine Grassmannian!</u>

$$\begin{array}{l} Z^{\nu,\nu',\lambda'} \\ \downarrow \phi \\ {}_{\nu'}Bun_B \end{array}$$ locally trivial fibration, with fiber $Gr_G^{\lambda'} \cap S^{\nu-\nu'+w_0(\lambda)}$ (U_K-orbit)

$$\begin{array}{l} Z^{\nu,\nu',\lambda'} \\ \downarrow \overrightarrow{'h} \\ {}_{\nu'}Bun_B \end{array}$$ locally triv fibrations with fibers $Gr_G^{-w_0\lambda'} \cap S^{-(\nu-\nu'+w_0(\lambda))}$

Facts 1. $\dim(S_\mu \cap Gr_G^\lambda) \leq \langle \mu+\lambda, \rho^\vee \rangle$
2. $H_c^{\langle \mu+\lambda, 2\rho \rangle}(S_\mu \cap Gr_G^\lambda, \mathbb{C}) = V^\lambda(\mu)$ (Satake)

- these give all necessary estimates.

Mirković proves this by induction ... on level of functions know leading term for q from classical Satake.

lower strata don't contribute to top cohomology - replace IC by $\mathbb{C}$..