11/12/03

[Drinfeld's conjectures

1. Description of $\mathcal{E}$-modules ~~[illegible]~~ concentrated on smooth Lagrangian submanifolds is valid (more or less) only in regular singularities assumption

2. "More or less" because I forgot about local monodromies ?] (think about case of curves)]

D. Gaitsgory : Geometrization of local Langlands

Start with global Langlands, geometrically & arithmetically
- impose Hecke eigencondition, look for solution

Local classical class field theory : "no formulation!"

K_x local field $x \in X$ curve

Characters with finite range $K_x^* \longrightarrow \mathbb{C}^*$

$\updownarrow$ match   } bijection

$\mathrm{Gal}(K_x) \longrightarrow \mathbb{C}^*$

- know this upstairs on characters factoring through $\mathbb{Z} = K^*/\mathcal{O}^*$ vs unramified Galois rep.

To extend : can impose compatibility with global CFT.

Alternatively : Brauer gp formulation via pairing into Brauer gp $= \mathbb{Q}/\mathbb{Z}$

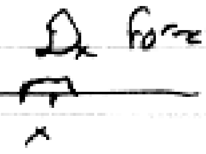
get map Abelianization of Gal group $\longrightarrow K^*$

Geometric local class field theory

Construction: $\rho : \mathrm{Gal}\, K_x \longrightarrow \mathbb{C}^*$

want to produce character of $K_x^* \to \mathbb{C}^*$.

Disregard Frobenius: characters of inertia $\longleftrightarrow$ chars of $\mathcal{O}^*$.

$X \ni x$ 

ρ 1-dim local system on X near x

Consider variety

$$\begin{tikzcd} Z := \{ x' \in X,\ \text{trivialization } \mathcal{O}(x')|_{D_x} \simeq \mathcal{O}_{D_x} \} \arrow[d] \\ X \end{tikzcd}$$

scheme of infinite type (projective limit of finite type schemes)

$G(\mathcal{Q}) = \mathcal{O}_x^*$ group scheme of automorphisms of trivial bdle $\mathcal{O}_{D_x}$

$\mathcal{O}_x^*$ acts on Z, Z is principal $\mathcal{O}_x^*$-bundle over X

Over open subset $x' \neq x$ this trivializes canonically!
$Z \times_X X \setminus x \simeq \mathcal{O}_x^* \times (X \setminus x)$ ... already have triv of $\mathcal{O}(x')$ over D_x for $x' \neq x$.

Construct sheaf $\underline{\mathbb{Q}_\ell} \boxtimes \rho$ on $\mathcal{O}_x^* \times (X \setminus x) = Z|_{X \setminus x}$
& now take nearby cycles
$\mathcal{F}_\rho = \psi(\mathbb{Q}_\ell \boxtimes \rho) \in \mathrm{Perv}(Z_x)$ ... → special fiber

.. in fact we have
<u>Lemma</u> 1. $\psi(\mathbb{Q}_\ell \boxtimes \rho)$ is in fact a 1-dim local system on Z_x.
[... will give rise to character $K_x^* \to \mathbb{C}^*$;]

2. $(\mathcal{O}_x^* \times Z_x \xrightarrow{act} Z)$ $act^* \mathcal{F}_\rho \simeq G_\rho \boxtimes \mathcal{F}_\rho$
where G_ρ again 1-dim local system on $\mathcal{O}_x^*$
- character sheaf! behaves multiplicatively
wrt multiplication on $\mathcal{O}_x^*$

⇒ produced G_ρ character sheaf on $\mathcal{O}_x^*$.
Base field = finite field ⇒ points of $\mathcal{O}_x^*$
now carry a <u>function</u>, which is desired character
of $\mathcal{O}_x^*$!
[Lang's construction of local CFT]

<u>Speculation</u> Formulate local geometric Langlands as
a construction of the following type:

ρ n-dimensional representation of $Gal(K_x)$
⇝ attach category C_ρ which carries an
action of group ind-scheme $GL_n(K_x)$ by functors

-- a "1-representation" of group instead of
rep of p-adic group --- geometrization!

naively expect C_ρ abelian category

C_ρ should give rise (for base field $\mathbb{F}_q$)
to a representation Π_ρ of $GL_n(K_x)(\mathbb{F}_q)$

.... roughly K-group $K(C_\rho)$ modulo some relations
of following form: ~~$\mathbb{F}_q$~~..
sheaves on affine Grassmannian $\longrightarrow$ functions
on affine Grassmannian is quotient of K_0 by
a lot... would be injective if remember not just
$\mathbb{F}_q$ but all its extensions...

Abelian story: category $C_\rho \simeq$ Vect so
has a distinguished object G_ρ.
C_ρ in abelian case = sheaves on $\mathcal{O}_x^*$ which
are equivariant against particular character:

$$C_\rho = \{ \mathcal{F} \in \mathrm{Perv}\, \mathcal{O}_x^* : m^* \mathcal{F} \simeq \mathcal{F} \boxtimes G_\rho \}$$

.... $\simeq$
Vect : every $\mathcal{F}$ is $G_\rho \otimes$(some vector space)

$\left[\cdots \text{really should extend from } \mathcal{O}_x^* \text{ to } \mathcal{O}_x^* \overset{\overset{\text{fiber}}{\shortparallel}}{\times} \mathbb{Z} = K_x^* \ldots \right]$

<u>Example</u> ρ = trivial local system

Take C_ρ = category of perverse sheaves on affine Grassmannian
with a Hecke eigenproperty, system of isomorphisms

$$\mathcal{F} * IC_\lambda \xrightarrow{\sim} \mathcal{F} \otimes V_\lambda \qquad V_\lambda \ G^\vee \text{ irrep space}$$

(live in ind-perverse sheaves! all have ∞-dim support!)

+ Need Compatibility ~~isomorphisms~~ $(\mathcal{F} * IC_\lambda) * IC_\mu \rightrightarrows \mathcal{F} \otimes V_\lambda \otimes V_\mu$
two isoms agree

G-action on category: look at <u>infinitesimally trivial</u> (char 0)
actions: Lie algebra $\mathfrak{g}$ acts trivially
.. like if G acts on X, $G \circlearrowright \mathcal{D}\text{-mod}(X)$
infinitesimally trivial

Example of object in $\mathcal{C}_\rho$:

$$\mathcal{F} = \bigoplus_\lambda IC_\lambda \otimes V_\lambda^* \qquad \text{"regular rep"}$$

or $\mathcal{F} * \mathcal{G}$ for any $\mathcal{G}$ with finite dim support

Q! What about the Frobenius?
-- this is trivial rep of inertia group -- looks like *generic* unramified representation... size of unramifieds varies, but this category $\mathcal{C}_\rho$ doesn't seem to feel Frobenius.

$\mathcal{C}_\rho$ is acted on by $G(K)$ by the translation action on Gr.

ρ carries action of all $G^\vee$ by automorphisms
$\Rightarrow$ expect $G^\vee$ to act by automorphisms on $\mathcal{C}_\rho$: $G^\vee$ acts on our Hecke eigensheaves $\mathcal{F}$ by changing isomorphism $\mathcal{F} * IC_\lambda \xrightarrow{\sim} \mathcal{F} \otimes \underset{\circlearrowright\, G^\vee}{V_\lambda}$ via action on reps $G^\vee$.

[ρ with nilpotent ramification $\Rightarrow$
answer by Bezrukavnikov]

$\mathcal{C}_\rho \circlearrowleft G(K)$, look at category of Iwahori-equivariant objects $\mathcal{C}_\rho^I$

Iwahori-equivariant category $D^I(\mathcal{F}\ell)$ acts

[expect $\mathcal{C}_\rho^I$ to be nonzero only if ρ has nilpotent monodromy, replace by I-monodromic wrt torus get other tame monodromies]

$$D^I(\mathcal{F}\ell) \circlearrowright \mathcal{C}_\rho^I$$

Bezrukavnikov-Arkhipov: construct a triangulated category family over nilpotent cone (all nilpotent ρ)

$$\begin{tikzcd} D(\mathcal{C}^I_{nilp}) \arrow[d] \\ Nilp/G^\vee \end{tikzcd}$$

$D(C^I_{nilp}) := D^I\left(G(K_x)/(N(K_x),\psi)\right)$ Whittaker sheaves

$\downarrow$

$Nilp/G^\vee$

I equivariant sheaves, r t $N(K_x)$-ψ equivariant

(pretend for now we can talk about sheaves which are $N(K_x),\psi$ equivariant -- "Whittaker" -- ∞ dim symmetry.
– work with local version

... analog of fns on groups which are right N,ψ -equivariant.

<u>Theorem</u>: this category $D(C^I_{nilp}) \simeq D_{G^\vee}(\tilde{N}^\vee)$ $``T^*G^\vee/B^\vee$
-- derived category of $G^\vee$-equiv coherent sheaves on resolution of nilpotent cone
--- does <u>NOT</u> preserve t-structures!

Have guess for <u>full</u> category $D(C_{nilp})$ rather than just nilpotent part – Frenkel Gaitsgory – in terms of critical level reps of Kac-Moody algebras.

Let ρ be a nilpotent local system, χ oper on formal punctured disc with underlying local system ρ

$C_\rho := (\hat{\mathfrak{g}}_{crit}\text{-mod})_\chi$ central character ρ

–

<u>Conjecture</u> For two oper structures χ, χ' on ρ the categories $(\hat{\mathfrak{g}}_{crit}\text{-mod})_\chi$, χ' are equivalent.

<u>$\widehat{G(K_x)} \curvearrowright \mathfrak{g}$-rep</u>:
M $\mathfrak{g}$-module – turn infinitesimally by group:

<u>infinitesimal rigidification</u> would identify these two: happens by fact that group acts infinitesimally on reps

-- but here need <u>level</u>, $\widehat{G(K)}$ acts

So $G \curvearrowright \mathfrak{g}$-reps is an example of infinitesimally trivial G-action on a category -- like $\mathcal{D}(G/B)$

Hope: construct a category $$\begin{tikzcd} C \arrow[d] \\ \mathrm{LocSys}_{G^\vee} \end{tikzcd}$$ specializing to $$\begin{tikzcd} C_\rho \arrow[d] \\ \{\rho\} \end{tikzcd}$$

Reduction of local system to Borel --- can attach distinguished automorphic sheaf, the Eisenstein series.. choosing reduction seems to rigidify question of automorphic sheaves, but not clear in Langlands philosophy...

Wakimoto modules --- again need to choose Borel on open to get category

<u>X global</u> $\pi_1(X - x) \xrightarrow{\rho_{glob}} {}^LG$

$\pi_1(K_x) \xrightarrow[\rho_{loc}]{} {}^LG$

$$\begin{tikzcd} \mathrm{Bun}_{G,x}(X) \arrow[d, dashed] \\ \text{trivialize on formal nbhd of } x \end{tikzcd}$$

$\rightsquigarrow$ category of perverse sheaves $\mathcal{F}$ on $\mathrm{Bun}_{G,x}(X)$ with Hecke structure on $X - x$

$$H^\lambda(\mathcal{F}) \xrightarrow{\sim} \mathcal{F} \boxtimes \rho^\lambda_{glob}|_{X - x}$$

Object of local category C_ρ should produce Hecke eigensheaf (global) in this category

In case of trivial local local system (unramified) get such a structure as desired

No level structure case: is Vect $\simeq$ automorphic sheaves on Bun_G with Hecke ??

On both sides have $G(K)$ actions, look in category of $G(\mathcal{O})$-equivariant objects... "multiplicity one" should mean an equivalence.

"Matrix coeffs": $V, W \in C \circlearrowleft G$

look at $R\mathrm{Hom}(g * V, W)$ sheaf on group, version of matrix coefficients

Inf. rigidification $\Rightarrow$ these carry $\mathfrak{g}$ action

Can "look for compact support" to rule out <u>trivial rep</u> (where get constant sheaf) e.g ~~[illegible]~~ Dr. on affine Grassmanian locally fin. generated $\mathcal{D}$-modules have this property! matrix coeffs are "finitely supported" ... a kind of cuspidality

Q: look at category of $\hat{\mathfrak{g}}$ modules at some level (e.g critical or strictly negative) with no integrable modules, does it satisfy this "cuspidality" property? matrix coeffs compactly supported? not true for integrable reps.

[Wants universal category over stack of Galois representations]

Consider Spec(Bernstein center) as analog of {Galois reps}

Generically to each point of LHS $\Rightarrow$ 1 irrep
cuspidals $\Longleftrightarrow$ isolated points on Spec(Bernstein center)
~~[illegible]~~
Consider category of $G(K_x)$ - 1-reps.
Is this roughly same as $\mathcal{O}$-modules on local systems? well first is a 2-category, second is a 1-category..
want C to have some such universality property among all $G(K_x)$ 1-reps:
– should automatically live over stack of
~~2-category with objects~~ local systems