C. Douglas – Twisted K-Theory of Lie Groups

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Twistings : – are bundles of spectra over a space X with fiber R $\Longrightarrow$ form twisted R-cohomology of X (as htpy classes of sections)

Classified by maps $X \longrightarrow BAut\,R$ homotopy automorphisms

$$\begin{array}{ccc} X & \longrightarrow & BAut\,R \\ & \searrow^{\text{elementary twisting}} & \cup \\ & & BGL_1 R \end{array}$$

for R E_∞ ... units in R

Motivations From elliptic cohomology:

- $X \xrightarrow[\tau]{} BGL_1 H\mathbb{C}$ twisting of $\mathbb{C}$-cohomology

$$\begin{array}{c} BGL_1 H\mathbb{C} \\ \text{is} \\ B\mathbb{C}^\times_{disc} \longrightarrow BS^1 \longrightarrow \mathbb{Z}\times BU \end{array}$$

end up with a K-theory class, namely a line bundle $\mathcal{L}(\tau)$. Twisted cohomology $H\mathbb{C}^*_\tau X = H\mathbb{C}^*(X;\mathcal{L}(\tau))$

... line bundle comes with flat connection

- K-theory twisting $X \xrightarrow{\tau} BGL_1 K \simeq T \times S \longrightarrow T$

with T a $K(\mathbb{Z},3)$ bundle over $K(\mathbb{Z}/2,1)$

$$\begin{array}{c} T \\ \downarrow \\ TMF \end{array}$$

... in some sense T generates TMF just as $\mathbb{C}P^\infty$ generates K-theory

So K-twist $\rightsquigarrow$ TMF class, in fact an "elliptic line bundle" $\mathcal{L}(\tau)$

eg
$$\begin{array}{ccc} K(\mathbb{Z},2) \longrightarrow P & & \\ \downarrow & & \\ X & \longrightarrow & K(\mathbb{Z},3) \end{array}$$

$\longleftrightarrow$ stack over X locally equivalent to $\left\{\begin{array}{c}\text{line} \\ \text{bundles}\end{array}\right\}$

$\longleftrightarrow$ a 2-line bundle $\in$ 2-vector bundle

$$K^*_\tau X = \text{``}K^*(X;\mathcal{L}(\tau))\text{''}$$

Physics D-branes ~ ∂-conditions in string theory
- charged in twisted K-homology K_*^τ (spacetime)
Twisting - Neveu-Schwarz H-flux field
Refinement: D-branes $\longrightarrow MSpin_*^{c,\tau} X \longrightarrow K_*^\tau X$
twisted Spin c - bordism

Computing ... Background: Given $\begin{matrix} F \to E \\ \downarrow \\ X \end{matrix}$ bundle of spectra over X

$$F^n(X) = \operatorname{colim} [X, \Omega^i F_{i+n}]$$
$$= \operatorname{colim} \Gamma_{hty}(X, X \times \Omega^i F_{i+n})$$ homotopy global sections

So can set $E^n(X) = \operatorname{colim} \Gamma_h(X, \Omega_X^i E_{i+n})$

Homology: $F_n(X) = \operatorname{colim} [S^{i+n}, (X \times F_i)/X]$
$E_n(X) = \operatorname{colim} [S^{i+n}, E_i/X]$

Note have canonical section always (basepoint) from X to E - so can collapse it

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Define $X \xrightarrow{\tau} K(\mathbb{Z},3) \Rightarrow K(\mathbb{Z},2) \to \begin{matrix} P(\tau) \\ \downarrow \\ X \end{matrix}$
$K(\mathbb{Z},2)$ bundle

$\Rightarrow$ bundle of spectra $K \longrightarrow \begin{matrix} P(\tau) \times_{K(\mathbb{Z},2)} K =: E_\tau \\ \downarrow \\ X \end{matrix}$

where $K(\mathbb{Z},2) \times K \longrightarrow K$
"$(L,V) \longmapsto L \otimes V$"

Def $K_\tau^* X := E_\tau^* X$

Alternate definition: $K_\tau^* X = [P(\tau), K]_{K(\mathbb{Z},2)}$ -equivalent

$\to$ generalize to twisted complexes of α-twisted sheaves

Primary tool! twisted Rothenberg–Steenrod sp. seq (Segal)

R-S spec sequence:

Given $S_\bullet : \Delta^{op} \to \mathcal{C}$ simplicial object

$$\Rightarrow \quad E^2 = H_p(E_q(S_\bullet)) \Longrightarrow E_{p+q}(|S_\bullet|)$$

[for category equipped with notion of homology...]

For us: $\mathcal{C} = \mathcal{K} = \left\{ \begin{array}{l} (X, E) \quad E \ K\text{-bundle}/X \\ \text{with fiberwise homotopy eq.} \end{array} \right.$

homology theory = twisted homology

e.g.: $\tau = k \in H^2(\Omega G; \mathbb{Z})$ (transgressed from $H^3(G,\mathbb{Z})$)

$\rightsquigarrow$ line bundle $\begin{array}{c} L^{-k} \\ \downarrow \\ \Omega G \end{array}$

Form $B_\tau \Omega G$ = twosided bar construction

$$B_\bullet(*, \Omega G, *_\tau)$$

where $\Omega G \times K \longrightarrow K$ (map $\Omega G \to pt$ in our category)

defines $*$

$$\Omega G \times K \xrightarrow[\tau \times id]{} K(\mathbb{Z},2) \times K \xrightarrow[act]{} K$$

is other map $\Omega G \to pt$, defining $*_\tau$ twisted pt.

Note $K_*(\Omega G \to *_\tau) = K_0 \Omega G \longrightarrow (K_0 *)_\tau$

homology in my category $\qquad c \longmapsto \langle L^k, c \rangle$

$|B_\tau \Omega G| = (G; E_\tau)$ bundle determined by twisting τ

Finally $E^2_{p,q} = \mathrm{Tor}^{K_\cdot \Omega G}_{pq}(K_\cdot *, (K_\cdot *)_\tau) \Rightarrow K^\tau_{p+q} G$

Example: G_2

1. K-homology of loop space: $K_\cdot \Omega G_2 = \mathbb{Z}[a,b,x] / a(a+3)-2b$

2. Representations for K-homology classes:

Bott generating variety $G_2/U(2) \xrightarrow{i} \Omega G_2$

generates homology, K-theory

$g \longmapsto g \ell g^{-1} \ell^{-1}$

where ℓ is loop given by ρ, perpendicular to it is $U(2)$

$$G_2/U_2 \supset SU(3)/SU(2) \supset SU(2)/U(1)$$
$$\quad \| \qquad\qquad \| \qquad\qquad \|$$
$$\quad V_3 \qquad\qquad V_2 \qquad\qquad V_1$$

$a = i_*[V_1] - 1 \qquad b = i_*[V_2] - 1, \qquad x = i_*[V_3] - 1$

3. Twisting map: $V = G/H \xrightarrow{i} \Omega G$

Compute $\langle L^k, i_*[V]\rangle = \langle i^* L^k, [V]\rangle = \langle ch(i^* L^k) \cup Td V, [V]_H\rangle$

$= \sum (-1)^i \,\mathrm{rk}\, H^i(V, i^* L^k)$ (Hirzebruch-Riemann-Roch)

$= \dim \mathrm{Irrep}_G(i^* L^k)$ (Bott holomorphic induction)

Result: $a \longmapsto (k+1) - 1 =: c_1 \qquad x \longmapsto \dfrac{(k-2)(k-2)(2k+3)(3k+4)(3k+5)}{120} =: c_3 - 1$

$b \longmapsto \binom{k+2}{2} - 1 =: c_2$

4. Tor calculation: $\mathrm{Tor}^{K\Omega G_2}(K_*, (K_*)_\tau) = \mathbb{Z}/\gcd(c_1, c_2, c_3)\langle y\rangle$

iterated filtration spectral seq

Theorem For G compact connected simple simply conn, rank n

$$K^\tau_*(G) = \Lambda[x_1, \ldots x_{n-1}] \otimes \mathbb{Z}/c(G,\tau)$$

$[K_* G = \Lambda[x_1, \ldots x_n]]$

Twisted Spin^c Bordism

Def $\quad MSpin_*^{c,\tau} X = \pi_i \left(P(\tau) \times_{K(\mathbb{Z},2)} MSpin^c \right)$

Remark: Index: $MSpin_*^{c,\tau} X \longrightarrow K_*^{\tau} X$

Model: $\left\{ \begin{array}{l} \text{bordism of oriented manifolds } M \\ \quad M \xrightarrow{i} X \;, \; c \in C^2(M) \text{ 2-cochain} \\ \quad \delta c = \beta w_2(\nu(M)) - i^* \tau \end{array} \right\}$

Bockstein second S-W class of normal bundle

Structure of twisted Roth.-Steenrod SS suggests manifold representatives for $MSpin_*^{c,\tau} X$

Mirror Symmetry

$$\begin{tikzcd}
\text{A-Branes} \arrow[r, leftrightarrow] \arrow[d, phantom, "\{"] & \subset \text{D-branes} \supset & \text{B-Branes} \arrow[d, phantom, "\{"] \\
\text{Lagrangian Submanifolds} \arrow[rr, leftrightarrow] \arrow[d] & \subset \text{Generalized } \mathbb{C}\text{-Submanifolds (Gualtieri)} \supset & \mathbb{C}\text{-submanifolds} \arrow[d] \\
\text{Fukaya category} \arrow[rr, leftrightarrow] & & \text{Derived category}
\end{tikzcd}$$

Generalized $\mathbb{C}$-submanifolds $\longrightarrow MSpin_*^{c,\tau} X \longrightarrow K_*^{\tau} X$

[note: $N \subset M$ with normal bundle K-oriented $\Longrightarrow$ pushforward $K^* N \longrightarrow K^* M$]