V. Drinfeld - Introduction to Eisenstein Series 11/11/99

$X/\mathbb{F}_q$ curve as usual, $F = \mathbb{F}_q(X)$, $\mathbb{A}$

Langlands philosophy: $\{$Representations of Weil $W_F \to GL(n, \overline{\mathbb{Q}}_\ell)\}$

$\updownarrow$

$\sim$ (approx.) $\{$irreducible automorphic representations of $GL(n, \mathbb{A})\}$ $C^\infty(GL(n,\mathbb{A})/GL(n,F))$

Simplest W_F reps: $\rho = \chi_1 \oplus \ldots \oplus \chi_n$, $\chi_i : W_F \to \overline{\mathbb{Q}}_\ell^*$

Maximal abelian quotient $W_F^{ab} \simeq \mathbb{A}^*/F^* \to \overline{\mathbb{Q}}_\ell^*$

$\rightarrow$ can forget about W_F in this case...

Corresponding automorphic representations: theory of principal Eisenstein series.

Nonprincipal Eisenstein series: $\rho = \rho_1 \oplus \ldots \oplus \rho_k$, $\sum_i \dim \rho_i = n$

π_i automorphic rep corresponding to ρ_i

$\rightsquigarrow$ try to construct rep corresponding to ρ.

General $G \supset T$, $\chi : T(\mathbb{A})/T(F) \to \mathbb{C}^*$ character

$\rightarrow$ construct automorphic representation of $G(\mathbb{A})$

[χ defines $W_F \to T^\vee \subset G^\vee$ $\rightsquigarrow$ should get $G(\mathbb{A})$ automorphic rep]

$\mathbb{C}$ = alg. closed field, char $\mathbb{C} = 0$. (eg. $\mathbb{C} = \overline{\mathbb{Q}}_\ell$!)

Local version $x \in X$ closed point, $\chi_x : T(F_x) \to \mathbb{C}^*$

$\rightsquigarrow$ expect $G(F_x) \to \mathrm{Aut}\, V$: just principal series $\mathrm{Ind}(\chi_x)$.

Thus fix $T \subset B \subset G$ $\chi_x : B(F_x) \to \mathbb{C}^*$

$\Rightarrow$ $V = \mathrm{Ind}_{B(F_x)}^{G(F_x)} \chi_x$. Unitary induced representation

So take global χ, restrict to each $T(F_x)$ $\Rightarrow$ unitary induced rep, take their tensor product

$\mathrm{Ind}\, \chi = \bigotimes_x \mathrm{Ind}\, \chi_x$ $G(\mathbb{A})$ rep.

— need to prove it is automorphic!

$\mathrm{Ind}\, \chi \subset C^\infty(G(\mathbb{A})/B(F)N(\mathbb{A}))$

- it is induced from $B(\mathbb{A})$, $\chi : B(\mathbb{A}) \to \mathbb{C}^*$, twist by ρ, and induce $\rightarrow$

functions on $G(\mathbb{A})$ transforming from right as χ under $B(\mathbb{A})$

— χ trivial on $N(\mathbb{A})$ and $T(F)$ $\Rightarrow$ $B(F)N(\mathbb{A})$

But we want $C^\infty(G(\mathbb{A})/G(F))$ $\to$ want equivariant linear operator $Eis' : C^\infty(G(\mathbb{A})/B(F)N(\mathbb{A})) \to C^\infty(G(\mathbb{A})/G(F))$

$$f \mapsto (Eis' f)(g) \equiv \sum_{\gamma \in G(F)/B(F)} f(g\gamma)$$

But $G(F)/B(F)$ is infinite.

$\boxed{\rightsquigarrow \text{analytic continuation of Eisenstein series}}$: formal series converges for some χ, continue to all characters (algebraically: certain formal series is actually rational!)

Suppose Eis' makes sense (at least on $\mathrm{Ind}\,\chi$).

Image should be Weyl group invariant (permuting χ_i ...)

Thus $w \in W$

$$\begin{tikzcd}
\mathrm{Ind}\,\chi \arrow[rr, "\sim", "\text{(from local theory)}"'] \arrow[dr, "Eis'_\chi"'] & & \mathrm{Ind}({}^w\chi) \arrow[dl, "Eis'_{w\chi}"] \\
& C^\infty(G(\mathbb{A})/G(F)) &
\end{tikzcd}$$

Need to show diagram commutes up to scalar (get some subspace of $C^\infty(G(\mathbb{A})/G(F))$)

- i.e. want $Eis'_{w\chi} = Eis'_\chi \cdot M(\chi)$ if chose identification
  some multiplier

$\boxed{\rightsquigarrow \text{functional equation for Eisenstein series}}$

Henceforth assume χ unramified, work only on $G(\mathcal{O}_{\mathbb{A}})$-left invariant functions.

$$\begin{tikzcd}
& G(\mathbb{A})/B(F) \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\
G(\mathbb{A})/B(F)N(\mathbb{A}) & & G(\mathbb{A})/G(F)
\end{tikzcd}$$

$Eis' = \pi_! \rho^*$ pull back & sum along fibers

ρ is proper - fibers $N(\mathbb{A})/N(F)$ compact (follows from case $N = \mathbb{G}_a$)

π has discrete, infinite fibers

$Eis' : C_0^\infty(G(\mathbb{A})/B(F)N(\mathbb{A})) \to C_0^\infty(G(\mathbb{A})/G(F))$ well defined

- functions with compact support.

Our functions never have compact support though - translation by $T(F)$ transforms them by character ....

$G(\mathcal{O}_A)\backslash G(A)/G(F) = Bun_G$

$G = \mathbb{G}_m$: $\quad G(\mathcal{O}_A)\backslash G(A) = \mathrm{Div}\, X$

$G(\mathcal{O}_A)\backslash G(A)/G(F) = \mathrm{Div}\, X/$ principal divisors

$D \mapsto \mathcal{O}_X(D)$ poles at most D - see identification

- have two identifications $-D$ or D but only one generalizes to other (nonabelian) groups.

$G(A) = G(\mathcal{O}_A)B(A)$ ~ Iwasawa decomp $G(F_x) = G(\mathcal{O}_x)B(F_x)$

$\Rightarrow \quad G(\mathcal{O}_A)\backslash G(A) = B(\mathcal{O}_A)\backslash B(A)$

$\Rightarrow \quad G(\mathcal{O}_A)\backslash G(A)/B(F) = B(\mathcal{O}_A)\backslash B(A)/B(F) = Bun_B$

$G(\mathcal{O}_A)\backslash G(A)/B(F)N(A) = B(\mathcal{O}_A)\backslash B(A)/B(F)N(A)$

$= T(\mathcal{O}_A)\backslash T(A)/T(F) = Bun_T$

$\longrightarrow$

$$\begin{tikzcd} & Bun_B \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ Bun_T & & Bun_G \end{tikzcd}$$

$Eis' = \pi_! p^*$, apply to f:

$f : G(\mathcal{O}_A)\backslash G(A) \to \mathbb{C}$, $\quad f(gtn) = \tilde{\chi}(t)^{-1} f(g)$ $\quad t \in T(A)$, $n \in N(A)$

$\tilde{\chi}(t) = \chi(t) \cdot \|t^{2\rho}\|^{\frac{1}{2}}$

$2\rho = \sum_{\alpha > 0} \alpha$ positive roots, $\quad 2\rho : T \to \mathbb{G}_m \quad t \mapsto t^{2\rho}$

$\Longleftrightarrow$

$f : Bun_T \to \mathbb{C}$ $\qquad$ [χ unramified $\Rightarrow$ $\chi : Bun_T \to \mathbb{C}^\times$]

$f(\mathcal{L}) = \chi(\mathcal{L})^{-1} \cdot q^{\langle \deg \mathcal{L}, \rho \rangle}$

$\deg \mathcal{L} \in \Gamma = \mathrm{Hom}(\mathbb{G}_m, T) = \mathrm{Hom}(T, \mathbb{G}_m)^*$

For each $\lambda : T \to \mathbb{G}_m$ and T-bundle $\mathcal{L}$ get degree of associated line bundle $\lambda_*(\mathcal{L})$

$\langle \deg \mathcal{L}, \rho \rangle := \deg \rho_*(\mathcal{L})$

Suppose Eis' makes sense — result will automatically be Hecke eigenfunction with known eigenvalues — follows since have construction before taking $G(\mathcal{O}_A)$-invariants, result after invariants autom. eigenfunction

$x \in X \Rightarrow \mathcal{H}(G(F_x), G(\mathcal{O}_x))$

- Change notation: $\mathrm{Eis}_\chi (\cancel{\ }) := \mathrm{Eis}\,(f_\chi)$

  $f_\chi : \mathrm{Bun}_T \to \mathbb{C} \qquad f_\chi(\mathcal{L}) := \chi(\mathcal{L}) \cdot q^{\langle \deg \mathcal{L}, \rho \rangle}$

  - replace χ^{-1} by χ to simplify notation.

Eis_χ is an $\mathcal{H}(G(F_x), G(\mathcal{O}_x))$-eigenfunction, with eigenvalue

[(as point of $\mathrm{Spec}\, \mathcal{H}(G(F_x), G(\mathcal{O}_x)) = (T(F_x)^* / T(\mathcal{O}_x)^*)/W$]

[($\wedge$: group of unramified characters)]

— eigenvalue is image of χ_x^{-1}.

L-functions $\chi : \mathrm{Pic}\, X \to \mathbb{C}^*$

$\boxed{L(\chi) := \sum_{D \geq 0} \chi(D)}$ $\qquad$ ($\chi(D)$ means $\chi([D] \in \mathrm{Pic}\, X)$))

$t \in \mathbb{C}^*$, $\nu_t : \mathrm{Pic}\, X \to \mathbb{C}^*$ defined by $\nu_t(a) = t^{\deg a}$

$L(\chi, t) = L(\chi \nu_t) = \sum_{D \geq 0} \chi(D)\, t^{\deg D}$ — makes sense as formal series in t.

$(t = q^{-s})$

When $\mathbb{C} = \mathbb{C}$ usual & $|t| < \frac{1}{q}$ this converges

Main result: this formal series is a rational function!

$\Rightarrow$ stick in $t=1$ get $L(\chi)$ .. if no pole at 1.

~ $L(\chi)$ rational function of χ :

$\chi \in \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{C}^*)$: $\qquad 0 \to \mathrm{Pic}^\circ X \to \mathrm{Pic}\, X \xrightarrow{\deg} \mathbb{Z} \to 0$

$0 \to \mathbb{C}^* \to \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{C}^*) \to \underbrace{\mathrm{Hom}(\mathrm{Pic}^\circ, \mathbb{C}^*)}_{\text{finite}} \to$

- thus $\mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{C}^*)$ is algebraic variety:

  $\mathbb{C}^*$-torsor over finite set

  $\longrightarrow$ notion of rational function - collection of rational fns on $\mathbb{C}^*$.

Eisenstein series again $\quad$ assume $G = GL_n$ for simplicity, $\mathcal{L} \in \mathrm{Bun}_{GL_n}$ rk n bundle

$$\mathrm{Eis}_\chi(\mathcal{L}) = \sum_{\substack{0 \subset \mathcal{L}_1 \subset \mathcal{L}_2 \subset \dots \mathcal{L} \\ \text{flags}}} \chi_1(\mathcal{L}_1)\, \chi_2(\mathcal{L}_2/\mathcal{L}_1) \dots \chi_n(\mathcal{L}_n/\mathcal{L}_{n-1}) \times$$
$$\times\, q^{\frac{1}{2} \sum_{i<j} (\deg \mathcal{L}_i/\mathcal{L}_{i-1} - \deg \mathcal{L}_j/\mathcal{L}_{j-1})}$$

$(\chi = (\chi_1, \dots \chi_n)$, $\chi_i : \mathrm{Pic}\, X \to \mathbb{C}^*)$

General G: $t \in T^\vee(\mathbb{C}) = Hom(\Gamma, \mathbb{C}^*)$

$\nu_t : Bun_T \xrightarrow{\deg} \Gamma \xrightarrow{t} \mathbb{C}^*$, $\quad \nu_t(\mathcal{L}) = t^{\deg \mathcal{L}}$

GL_n: $$Eis_{\chi\cdot\nu_t}(\mathcal{L}) = \sum_{0\subset \mathcal{L}_1\subset\dots\subset\mathcal{L}} \chi_1(\mathcal{L}_1)\chi_2(\mathcal{L}_2/\mathcal{L}_1)\dots\chi_n(\mathcal{L}/\mathcal{L}_{n-1})\, q^{\sum\cdots} \times$$
$$\times\, t_1^{\deg \mathcal{L}_1}\, t_2^{\deg \mathcal{L}_2/\mathcal{L}_1}\dots t_n^{\deg \mathcal{L}/\mathcal{L}_{n-1}}$$
$$= \sum_{0\subset\mathcal{L}_1\subset\dots\subset\mathcal{L}} \chi_1(\mathcal{L}_1)\dots\chi_n(\mathcal{L}/\mathcal{L}_{n-1})\, q^{\sum} \cdot \left(\frac{t_1}{t_2}\right)^{\deg \mathcal{L}_1}\dots \left(\frac{t_{n-1}}{t_n}\right)^{\deg\mathcal{L}_{n-1}} t_n^{\deg\mathcal{L}}$$

- If we fix all degrees of $\mathcal{L}_i$ get <u>finite</u> sum: finitely many flags in given $\mathcal{L}$ with fixed degrees.
  In fact fixing $\deg \mathcal{L}_i/\mathcal{L}_{i-1} \Rightarrow$ only finitely many $\mathcal{L}$, flags:
  fixed degree $\Rightarrow$ finitely many line bundles $\mathcal{L}_i/\mathcal{L}_{i-1}$ possible
  $Ext(\mathcal{A}, \mathcal{B})$ always finite dimension vector space $\Rightarrow$ finite set ($\mathbb{F}_q$ finite)
  $\Rightarrow$ well defined formal series.

- $\mathcal{L}$ fixed $\Rightarrow \deg \mathcal{L}_i \le M_i(\mathcal{L})$ bounded from above:
  $\dim H^0(\mathcal{L}) \geq \dim H^0(\mathcal{L}_i) \geq \deg\mathcal{L}_i + i(1-g)$

Hence $$Eis_{\chi\nu_t}(\mathcal{L}) \in \mathbb{C}[[\tfrac{t_2}{t_1},\dots,\tfrac{t_n}{t_{n-1}}]] \otimes_{\mathbb{C}[\frac{t_2}{t_1},\dots,\frac{t_n}{t_{n-1}}]} \mathbb{C}[t_1^{\pm1},\dots,t_n^{\pm1}]$$

– makes sense for any G: $\frac{t_{i+1}}{t_i} = t^{-\alpha_i^\vee}$ simple coroots, & $\mathbb{C}[t_1^{\pm1},\dots t_n^{\pm1}] = \mathbb{C}[T^\vee]$.

Thus as function of χ, $Eis_{\chi\nu_t}(\mathcal{L})$ is function on

$$\begin{tikzcd} & T^\vee \times_{\mathbb{A}^r} \hat{\mathbb{A}}^r \arrow[dl] \arrow[dr] & \\ T^\vee \arrow[dr, "{(-\alpha_1^\vee,\dots,-\alpha_r^\vee)}"'] & & \hat{\mathbb{A}}^r \text{ completion at } 0 \arrow[dl] \\ & \mathbb{G}_m^r \hookrightarrow \mathbb{A}^r \ni 0 & \end{tikzcd}$$

Morally Eis converges for $\left|\frac{t_{i+1}}{t_i}\right| \ll 1$ ....

Note Weyl chambers: $\left|\frac{t_{i+1}}{t_i}\right| \le 1$ ....
In fact for $\mathbb{C} = \mathbb{C}$ complex numbers,
Eis converges for $\left|\frac{t_{i+1}}{t_i}\right| < q^{-1}$.

Theorem 1 The formal series $\mathrm{Eis}_{\chi\cdot\nu_t}(\mathcal{L})$ is a rational function of $t \in T^\vee(\mathbb{C})$ ($\mathcal{L} \in \mathrm{Bun}_G$ fixed).

Thus Eis_χ makes sense as a rational function of $\chi \in \mathrm{Hom}(\mathrm{Bun}_T, \mathbb{C}^*)$ (natural structure of algebraic variety: finitely many copies of $T^\vee$)

Consider $\mathrm{Eis}^L_\chi := \mathrm{Eis}_\chi \cdot \prod_{\substack{\check\alpha<0 \\ \text{neg coroots} \in \Gamma}} L(\chi^{\check\alpha}\cdot \nu_q^{-1})$, rational function

($\chi: \mathrm{Bun}_T \to \mathbb{C}^*$ & $\chi^{\alpha^\vee}: \mathrm{Bun}_{\mathbb{G}_m} \xrightarrow{\alpha^\vee} \mathrm{Bun}_T \xrightarrow{\chi} \mathbb{C}^*$ character of Pic)

$\nu_q^{-1}(A) = q^{-\deg A}$ character of Pic.

$\widetilde{\mathrm{Eis}}^L_\chi := \mathrm{Eis}^L_\chi \cdot \chi^{\check\rho}(\omega_X)$ $\check\rho = \frac{1}{2}$ sum of pos. coroots, $\omega_X \in \mathrm{Pic}\, X$ canonical class

(simple formulation) **Theorem 2** (1) $\widetilde{\mathrm{Eis}}^L_{w\chi} = \widetilde{\mathrm{Eis}}^L_\chi$ for $w \in W$

(correct formulation) (2) $\mathrm{Eis}^L_{w\chi} = \mathrm{Eis}^L_\chi \cdot \chi^{\check\rho - w^{-1}(\check\rho)}(\omega_X)$

Problem: $\widetilde{\mathrm{Eis}}$ not quite well defined ... $2\check\rho \in \Gamma$, $2\check\rho: \mathbb{G}_m \to T$
$\rightsquigarrow$ well defined $\chi^{2\check\rho}$.
So $\chi^{\check\rho}$ well defined if $\check\rho \in \Gamma$, or if we fix $\omega_X^{\frac{1}{2}}$.
(over finite field $\omega_X^{\frac{1}{2}}$ may not even exist...)

$\boxed{\text{Note:}}$ automorphic sheaf should live on Bun_G only if fix square root of ω_X — truly lives on moduli of <u>twisted</u> G-bundles.
$G = SL(2)$: rank 2 bundle $\mathcal{L}$, $\Lambda^2\mathcal{L} \xrightarrow{\sim} \omega_X$
<u>not</u> $\mathcal{O}_X$ · isomorphic if have choice of $\omega_X^{\frac{1}{2}}$.
— that is why we need square roots, $\check\rho$ ...

Theorem 3 The only poles of $\mathrm{Eis}^L_\chi(\mathcal{L})$ are simple poles at hypersurfaces $\chi^{\alpha^\vee} = \nu_q$, $\alpha^\vee \in$ ~~simple~~ all coroots

Usual ζ-function: poles at $s = 0, 1 \Leftrightarrow t = 1, \frac{1}{q}$.
- here have two poles in each direction

Def. χ is <u>generic</u> if for each $\alpha^\vee$, $\chi^{\alpha^\vee}|_{\mathrm{Pic}^0}$ is nontrivial
$\chi^{\alpha^\vee}: \mathrm{Pic}^0 \to \mathbb{C}^*$. (depends only on connected component)

$\Rightarrow$ χ generic have no poles.
GL_n : $\exists$ just need $\chi_i|_{\mathrm{Pic}^0} \neq \chi_j|_{\mathrm{Pic}^0}$

Geometric case - restrict to this setting - hard otherwise

$\boxed{\text{Reduction to rank 1}}$ assume $G = GL(n)$ for simplicity
Assume know all three for $GL(2)$ ... functional equation via formal arguments (ignoring infinities):

$$Eis_\chi(\mathcal{L}) = \sum_{0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{n-1} \subset \mathcal{L}} \chi_1(\mathcal{L}_1) \cdots \chi_n(\mathcal{L}_n/\mathcal{L}_{n-1})\, q^{\frac{1}{2}\Sigma}$$

(enough to work with simple parabolics)

$$= \sum_{0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{i-1} \subset \mathcal{L}_{i+1} \subset \cdots \subset \mathcal{L}} a(\mathcal{L}, \mathcal{L}_1, \ldots \mathcal{L}_{i-1}, \mathcal{L}_{i+1} \ldots \mathcal{L}_n) \cdot \sum_{\substack{\mathcal{L}_i \\ (\mathcal{L}_{i-1} \subset \mathcal{L}_i \subset \mathcal{L}_{i+1})}} q^{\left(\frac{1}{2} \deg \mathcal{L}_i/\mathcal{L}_{i-1} - \deg \mathcal{L}_{i+1}/\mathcal{L}_i\right)} \cdot \chi_i(\mathcal{L}_i/\mathcal{L}_{i-1})\, \chi_{i+1}(\mathcal{L}_{i+1}/\mathcal{L}_i)$$

[expression a is independent of χ_i, χ_{i+1}]

RHS is Eisenstein for GL_2: $Eis_{\chi_i \chi_{i+1}}(\mathcal{L}_{i+1}/\mathcal{L}_{i-1})$ rank two bundle

$\Longrightarrow$ functional equation formally follows from $GL(2)$

- must take L's into account also, $\prod_{r>s} L(\frac{\chi_r}{\chi_s} q^{-1})$
- Splits off the same way:
  term outside GL_2 term is symmetric in χ_i, χ_{i+1}.

<u>Honest proof</u> Fix χ, assume trivial on connected component of unit group - start from trivial χ and multiply

- Show $Eis_{\chi t}$ rational, symmetric in t & with prescribed poles at $\frac{t_i}{t_j} = q$ — $f(t) = \prod_{i \neq j} (\frac{t_i}{t_j} - q)$
- Must show $f(t) Eis_{\chi t}(\mathcal{L}) \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$

$$\overset{\shortparallel}{\sum_\lambda c_\lambda t^\lambda} \qquad \lambda \in \mathbb{Z}^n = \Gamma$$

Consider $Supp = \{\lambda \in \Gamma \mid c_\lambda \neq 0\}$ - want to show it is finite.

(1) $Supp \subset F + \sum_i \mathbb{Z}_- \alpha_i^\vee$, F finite set

- clear since $\sum c_\lambda t^\lambda \in \mathbb{C}[[\frac{t_2}{t_1} \ldots \frac{t_n}{t_{n-1}}]] \underset{\mathbb{C}[\frac{t_2}{t_1}, \ldots \frac{t_n}{t_{n-1}}]}{\otimes} \mathbb{C}[t_1^{\pm 1} \ldots t_n^{\pm 1}]$

(2) $\forall w \in W$ $w(Supp) = Supp + \lambda_w$, $\lambda_w \in \Gamma$

Lemma $\mathrm{Eis}_\chi^L(\mathcal{L}) = \sum_{\substack{A \subset \mathcal{L} \\ \text{Subsheaves}}} \chi_1(A)\,\chi_2\left(\cancel{\det \mathcal{A}}\ \frac{\det \mathcal{L}}{A}\right) q^{\frac{1}{2}(\deg A - \deg \frac{\det \mathcal{L}}{A})}$

[L-factor allows to write as sum over subsheaves instead of subbundles
Note $A \subset \mathcal{L}$ maximal $\Rightarrow \mathcal{L}/A = \frac{\det \mathcal{L}}{A}$.
$\to$ L function is ratio of these two expressions ($\rightsquigarrow$ attempts to prove Riemann hypothesis)]

Proof $A \subset A' \subset \mathcal{L}$ maximal subsheaf ...
$A = A'(-D)$ $\quad D \geq 0$ ...
Can replace A by pair (A', D), A' maximal.

$$\text{RHS of lemma} = \sum_{\substack{0 \subset A' \subset \mathcal{L} \\ \text{Subbundles}}} \sum_{D \geq 0} \chi_1(A')\,\chi_2(\mathcal{L}/A')\, q^{\frac{1}{2}(\deg A' - \deg \mathcal{L}/A')}$$

$$\left(\chi_1(A) = \frac{\chi_1(A')}{\chi_1(D)} \text{ etc.} \right) \qquad \cdot \frac{\chi_2}{\chi_1}(D)\, q^{-\deg D}$$

$$= \left(\sum_{A' \subset \mathcal{L}} \chi_1(A')\chi_2(\mathcal{L}/A') q^{\cdots} \right) \cdot \left(\sum_D \frac{\chi_2}{\chi_1}(D)\, q^{-\deg D} \right) \quad \blacksquare$$

$$\mathrm{Eis}_\chi^L(\mathcal{L}) = \sum_{\substack{A \subset \mathcal{L} \\ \text{subsheaf}}} \chi_1(A)\,\chi_2\left(\frac{\det \mathcal{L}}{A}\right) q^{\frac{\cdots}{2}}$$

(Proof analogous to proof of rationality & func. eqn of ζ-fn over finite field)

$$= \sum_{A \in \mathrm{Pic}\, X} \chi_1(A)\,\chi_2\left(\frac{\det \mathcal{L}}{A}\right) q^{\frac{2\deg A - \deg \mathcal{L}}{2}}$$

$$\#\left(\mathrm{Hom}(A, \mathcal{L}) \setminus \{0\} / \mathbb{F}_q^*\right) \qquad \text{number of classes of } A \hookrightarrow \mathcal{L}$$

$$= \sum_{\mathrm{Pic}\, X} \chi_1(A)\,\chi_2\left(\frac{\det \mathcal{L}}{A}\right) q^{[\ ]} \cdot \frac{q^{\dim \mathrm{Hom}(A, \mathcal{L})} - 1}{q - 1}$$

Use remarkable formula $\sum_{n \in \mathbb{Z}} z^n = 0$:

$$\sum_{n \in \mathbb{Z}} z^n = \underbrace{\sum_0^\infty z^n}_{\text{converges near } 0} + \underbrace{\sum_{-1}^{-\infty} z^n}_{\text{converges near } \infty} = \frac{1}{1-z} + \frac{z^{-1}}{1-z^{-1}} = 0$$

(really δ-function at 0 ...)

Use it for formal proof .... allows us to drop $\frac{q^n - 1}{q-1}$

- follows from functional equation: showed it's symmetric wrt permuting t_i, t_{i+1} in naive sense
  - Suppose $F(t) \in \mathbb{C}(t)$ $F(t) = F(t^{-1})$

look at Laurent expansion $F(t) = \sum a_n t^n$ at $t=0$
$\not\Rightarrow$ $a_n = a_{-n}$: equation compares Laurent coefficients at $0, \infty$.
BUT answer is obviously yes for Laurent polynomial.

We've killed all poles assuming they are where we think they are
- but know this for $GL_2 \Rightarrow$ for simple reflections..
  shift λ_w comes from monomials in functional equation....

(1), (2) $\Rightarrow$ Supp is finite. Hence Thms 1,2,3 reduce to $GL(2)$.

Part II 11/18

$G = GL(2) \supset T$, $\quad Bun_T = Pic X \times Pic X$

$\chi : Bun_T \to \mathbb{C}^* \iff \chi = (\chi_1, \chi_2)$, $\chi_i : Pic X \to \mathbb{C}^*$

$\mathcal{L} \in Bun_{GL(2)}$

$$Eis_\chi^L(\mathcal{L}) := L\left(\frac{\chi_2}{\chi_1} \cdot \nu_q^{-1}\right) \cdot \sum_{A \subset \mathcal{L}} \chi_1(A)\chi_2(\mathcal{L}/A)\, q^{\frac{1}{2}(\deg A - \deg \mathcal{L}/A)}$$

$\nu_t : Pic X \to \mathbb{C}^*$, $\nu_t(A) = t^{\deg A}$ $\quad A$ line bundle

$Eis^L_{\chi_1, \chi_2 t}(\mathcal{L})$ function of t : add factor $t^{\deg \mathcal{L}/A}$
makes sense as formal series in t, $\in \mathbb{C}((t))$

Theorem
- $Eis^L_{\chi_1, \chi_2 \nu_t}(\mathcal{L}) \in \mathbb{C}(t)$ rational $\Rightarrow$ setting $t=1$ get $Eis^L_\chi(\mathcal{L})$ as rational function of χ.
- Only poles are simple poles at $\chi_1/\chi_2 = \nu_q^{\pm 1}$
- $Eis^L_{\chi_2, \chi_1} = Eis^L_{\chi_1, \chi_2} \cdot \frac{\chi_1}{\chi_2}(\omega_X)$

L analog of Dirichlet L-fns for $\mathbb{Q}$ for function fields
χ nonramified $\Rightarrow$ character of Pic
$L(\chi) = \sum_{D \geq 0} \chi(D)$ nonnegative divisors

introduce t + prove rational fn in t.

- insert formal parameter t, use identity for formal series

$$(q-1)\cdot \mathrm{Eis}_{\vec{\chi}}(\mathcal{L}) = \sum_{A\in \mathrm{Pic}\, X} \chi_1(A)\,\chi_2\!\left(\frac{\det \mathcal{L}}{A}\right) q^{\frac{2\deg A - \deg \mathcal{L}}{2}} \cdot q^{\dim H^0(A^{-1}\mathcal{L})}$$

$(q-1)$ factor: should sum not over Pic variety but over <u>stack</u> $\mathrm{Pic}/\mathbb{F}_q^*$!

Riemann-Roch: $\dim H^0(A^{-1}\mathcal{L}) - \dim H^0(\omega_X \otimes A \otimes \mathcal{L}^*)$
$= \deg(A^{-1}\mathcal{L}) + 2 - 2g$.

$\mathcal{L}^* \simeq \mathcal{L}/\det \mathcal{L}$ (rank two bundle)

$\Rightarrow \dim H^0(A^{-1}\mathcal{L}) - \dim H^0(\tilde{A}^{-1}\otimes \mathcal{L}) = \deg(A^{-1}\mathcal{L}) + 2 - 2g$.

where $\tilde{A} = \dfrac{\det \mathcal{L}}{\omega_X\, A}$

<u>Claim</u> Change of variables $A \longrightarrow \tilde{A}$ exactly gives other term in functional equation!

- Can make rigorous! δ-function contributes the poles of the Eisenstein series..

(rationality: for deg A big no subbundles $\to$ get explicit tail in sum arithmetic progression!)

<u>Geometrization</u> —

Set $\mathbb{C} = \overline{\mathbb{Q}}_\ell$. $\chi_1, \chi_2 : \mathrm{Pic}\, X \longrightarrow \overline{\mathbb{Q}}_\ell^*$.

- By class field theory, $\chi_i : \pi_1(X) \longrightarrow \overline{\mathbb{Q}}_\ell^*$
  $\rightsquigarrow$ local system $\mathcal{E}_i$ on X..
- $\mathrm{Pic}\, X = \underline{\mathrm{Pic}\, X}(\mathbb{F}_q)$ Picard scheme.
  $\rightsquigarrow$ would like a sheaf on $\underline{\mathrm{Pic}\, X}$, $\mathrm{tr}(Fr$ on stalks$) = \chi_i$.

[<u>Theorem</u> A rank 1 local system $\mathcal{E}$ on X canonically extends to a rank 1 local system $\underline{\mathcal{E}}$ on $\underline{\mathrm{Pic}}\, X$.

$(X \xrightarrow[A\text{-}J]{} \underline{\mathrm{Pic}}^1 X \subset \mathrm{Pic}\, X)$

$\underline{\mathcal{E}}$ should have properties of character: $\chi(ab) = \chi(a)\chi(b)$

$\mu: G\times G \to G$

Def A <u>character</u> on a commutative algebraic group G is a rank 1 local system $\mathcal{M}$ on G, equipped with isomorphisms $\mathcal{M}_g \otimes \mathcal{M}_h \xrightarrow{\sim} \mathcal{M}_{gh}$ ($g,h\in G$, $\mathcal{M}_g$: fiber) satisfying:

1. Commutative & associative [i.e. $\bigcup_{g\in G}(\mathcal{M}_g \setminus \{0\})$ is commutative group, an extension of G by $\overline{\mathbb{Q}}_\ell^*$]
2. they isomorphisms come from an isomorphism $\mathcal{M}\boxtimes\mathcal{M} \xrightarrow{\sim} \mu^*\mathcal{M}$

[In particular fiber over $1 \simeq \overline{\mathbb{Q}}_\ell^*$ canonically]

Above theorem may be reformulated: $\mathcal{E}$ on X uniquely extends to a character $\underline{\mathcal{E}}$ on $\underline{\mathrm{Pic}}\, X$.

<u>Construction</u> (Deligne): $X^{(n)} = \{\text{eff. divisors of deg } n\} = X^n/S_n$ $\quad p: X^n \to X^{(n)}$

$\pi: X^{(n)} \to \mathrm{Pic}\, X$

$\mathcal{E} \rightsquigarrow \mathcal{E}^{(n)} = (p_* \mathcal{E}^{\boxtimes n})^{S_n}$

$\mathcal{E}$ rank 1 $\to$ local system with fiber $\mathcal{E}^{(n)}_{x_1+\dots+x_n} = \mathcal{E}_{x_1}\otimes\dots\otimes\mathcal{E}_{x_n}$.

Suppose $\underline{\mathcal{E}}$ is character extending $\mathcal{E}$

$\underline{\mathcal{E}}_{x_i} = \mathcal{E}_{x_i}$, so $\mathcal{E}_{x_1}\otimes\dots\otimes\mathcal{E}_{x_n} = \underline{\mathcal{E}}_{x_1+\dots+x_n}$

$\pi_n: X^{(n)} \to \mathrm{Pic}^n X$. Want $\pi_n^* \underline{\mathcal{E}} = \mathcal{E}^{(n)}$

.. Suppose $n \geq g$, π_n epimorphism ..

For any n fibers are (when nonempty) projective spaces.

Restriction of $\mathcal{E}^{(n)}$ to fiber is local system on $\mathbb{P}^k \Rightarrow$ trivial $\to \mathcal{E}^{(n)} = \pi_n^* \mathcal{F}_n$, $\mathcal{F}_n$ on $\mathrm{Pic}^n X$.

Want $\underline{\mathcal{E}}|_{\mathrm{Pic}^{n\geq g} X} = \mathcal{F}_n \to$ determines $\underline{\mathcal{E}}$ by multiplicativity. ∎

<u>Geometric Eisenstein Series</u> (after Laumon):

$G = GL_2$. X/alg closed field

$\mathcal{E}_1, \mathcal{E}_2$ rank 1 local systems on X.

Want to construct $\mathrm{Eis}_{\mathcal{E}_1,\mathcal{E}_2}$ on $\mathrm{Bun}_{GL(2)}$.

$\mathrm{Bun}_B \subset \overline{\mathrm{Bun}}_B = \{$ rank two bundle $\mathcal{L}$, rank 1 subsheaf $A \}$

$$\begin{array}{ccc} & \overline{Bun}_B & \\ \overset{q}{\swarrow} & & \overset{p}{\searrow} \\ Bun_T = Pic X \times Pic X & & Bun_{GL(2)} \end{array}$$

$q(\mathcal{A}, \mathcal{L}) = (\mathcal{A}, \det \frac{\mathcal{L}}{\mathcal{A}})$

<u>Def</u> $Eis_{E_1, E_2} \doteq Rp_! \, q^*(E_1 \otimes E_2) \{ \dim \overline{Bun}_B \} \otimes (E_2 E_1^{-1})_{\omega_X^{1/2}}$

analog of $q^{\frac{1}{2}(\dots)}$ — analog of $\chi_2/\chi_1(\omega_X^{\frac{1}{2}})$ in modified functional equation

Which $\omega_X^{\frac{1}{2}}$: have to choose one, or better work on twisted version of Bun_G.

$\mathcal{F}$ = complex of sheaves, $\mathcal{F}\{k\} := \mathcal{F}[k](\frac{k}{2})$ → Tate twist

shift dimension by k

Operation $\mathcal{F} \to \mathcal{F}\{k\}$ very natural:
IC sheaf on smooth manifold not simply constant sheaf $\overline{\mathbb{Q}}_\ell$ but rather $\overline{\mathbb{Q}}_\ell\{k = \dim(\text{manifold})\}$

– Tate & shift both occur in Verdier duality.

$k = \deg \mathcal{A}$
$l = \deg \mathcal{L} - \deg \mathcal{A}$

$\overline{Bun}_B^{k,l}$ smooth connected stack of dimension $l - k + 3g - 3$.
– need to multiply Eisenstein by something like q^{3g-3} for naturality

---

- $q^*(E_1 \otimes E_2)\{\dim \overline{Bun}_B\}$ is <u>perverse</u>, and pure of weight 0 if E_1, E_2 are:
  $\overline{Bun}_B^{k,l}$ is smooth, this is local system – just need to get weight & shift right

- p is "almost" proper: $\overline{Bun}_B$ has ∞ many connected components, but $p_{kl}: \overline{Bun}_B^{k,l} \to Bun_{GL_2}$ is <u>proper</u>.

  ⇒ can replace $p_!$ by p_*. (up to difference between direct product & direct sum ⟷ ∞ many coeffs of Eisenstein series.. look at each separately → replace $(\oplus)^*$ by a direct sum instead of product..

Then this construction is "self-dual" under Verdier

$\mathbb{D}\, Eis_{\mathcal{E}_1, \mathcal{E}_2} = Eis_{\mathcal{E}_1^{-1}, \mathcal{E}_2^{-1}}$:

p proper $\Rightarrow$ $\mathbb{D} p_! = p_* \mathbb{D}$.

((classical analog of this: Verdier has no obvious analog for functions ⤳ something like complex conjugation...)

Why is p proper: fibers of $\overline{Bun}_B^{k,l} \to Bun_{GL(2)}$ are projective schemes (weaker than properness)

⤳ note p representable: fiber over $\mathcal{L}$ is all rank 1 subsheaves of $\mathcal{L}$, no automorphisms.

{rk 1 subsheaves of $\mathcal{L}$} is projective: I. fix $\mathcal{A} \in Pic^k X$

$\{\mathcal{A} \hookrightarrow \mathcal{L}\} \in \mathbb{P}(Hom(\mathcal{A}, \mathcal{L}))$, and set of all $\mathcal{A}$'s is a projective variety $\to$ projective scheme

II. (more scientific explanation) For any vector bundle $\mathcal{L}$ of rank n

$\{\mathcal{A} \subset \mathcal{L}\}$ all subsheaves of rank k and degree d is a projective scheme (Grothendieck)

- Quot scheme construction (quotients: easier to formulate families of quotients...) $Quot_{d,k} \mathcal{L}$ = scheme parametrizing quotients of deg d, rank $n-k$. This is projective.

Theorem 1 $Eis_{\mathcal{E}_1, \mathcal{E}_2} = Eis_{\mathcal{E}_2, \mathcal{E}_1}$ if $\mathcal{E}_1 \not\cong \mathcal{E}_2$: (functional equation)

[classical analog: Eis has no pole at $\mathcal{E}_1 \neq \mathcal{E}_2$ ...]

- not understood what happens at $\mathcal{E}_1 \simeq \mathcal{E}_2$!

Proof parallel to classical setting

"equality": canonical isomorphism functorial in $\mathcal{E}_1, \mathcal{E}_2$

- $\mathbb{G}_m$ acts on $\mathcal{E}_1$ & on $\mathcal{E}_2$: above iso preserves bigrading ($\mathbb{G}_m \times \mathbb{G}_m$ action)

$Eis^{k,l}_{\mathcal{E}_1, \mathcal{E}_2} = Eis^{l,k}_{\mathcal{E}_2, \mathcal{E}_1}$ ... FALSE for $\mathcal{E}_1 \simeq \mathcal{E}_2$

(unlike duality statement)

$f(t) = \sum_{-N}^{\infty} a_n t^n \in \overline{\mathbb{Q}}_\ell(t)$, $f(t) = f(t^{-1}) \not\Rightarrow a_n = a_{-n}$

- true for Laurent polynomials ...

But when $\mathcal{E}_1 = \mathcal{E}_2$ don't have Laurent polynomials so don't get equality of coefficients $\mathrm{Eis}^{k,l}$ . . .
Geometrically must understand what happens when $\mathcal{E}_1 \rightsquigarrow \mathcal{E}_2$ approach.

Theorem 2 (Exercise) $\mathrm{Eis}_{\mathcal{E}_1, \mathcal{E}_2}$ is a Hecke eigensheaf with eigenvalue $\mathcal{E}_1 \oplus \mathcal{E}_2$

Proof: in classical setting proof ~~easy~~ for easiest Hecke operator $T_x \longleftrightarrow$ 2-dim rep of GL_2, easy. Translates almost immediately to geometry.

Theorem For $\mathcal{E}_1 \not\cong \mathcal{E}_2$, $\mathrm{Eis}_{\mathcal{E}_1, \mathcal{E}_2}$ is perverse, & its restriction to each connected component is irreducible.

Geometric Eisenstein Series III

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$G = GL(2)$ $\quad \mathrm{Bun}_T \xleftarrow{q} \overline{\mathrm{Bun}}_B \xrightarrow{p} \mathrm{Bun}_G$. $\mathcal{E}_1, \mathcal{E}_2$ rk 1 loc sys$/X \Rightarrow \underline{\mathcal{E}_1}, \underline{\mathcal{E}_2}$ on Bun_{GL_1} (characters)
$\Rightarrow Rp_! q^* (\underline{\mathcal{E}_1} \boxtimes \underline{\mathcal{E}_2}) \{\dim \overline{\mathrm{Bun}}_B\} \otimes \left(\frac{\underline{\mathcal{E}_1}}{\underline{\mathcal{E}_2}}\right)_{\omega_X^{\frac{1}{2}}}$

General G: – for GL_n have Laumon compactification of Bun_B, which is smooth (as for GL_2).
– all reductive groups: (Drinfeld) compactification, not smooth.

$\underline{\mathcal{E}_1}, \underline{\mathcal{E}_2} \longleftrightarrow T^\vee$ local system $\mathcal{E}$ in general.
$\longrightarrow$ character $\underline{\mathcal{E}}$ on Bun_T

$\overline{\mathrm{Bun}}\, B \supset \mathrm{Bun}_B$
$q \downarrow \quad \searrow p$
$\mathrm{Bun}_T \qquad \mathrm{Bun}_G$

1) $G = GL_n$, have $\overline{\mathrm{Bun}}_B^{\text{Laum}}$ smooth

$\mathrm{Eis}_{\mathcal{E}} = Rp_! q^* \underline{\mathcal{E}} \{\dim \overline{\mathrm{Bun}}_B\} \otimes (\underline{\mathcal{E}})_{(\omega_X^{\frac{1}{2}})^{2\check{\rho}}}$

$2\check{\rho}: \mathbb{G}_m \longrightarrow T$, $\omega_X^{\frac{1}{2}} \in \mathrm{Bun}_{\mathbb{G}_m}$ . . . .

– this definition is self-dual under Verdier (due to smoothness of $\overline{\mathrm{Bun}}_B^{\text{Laum}}$, properness)

2) General G: $\overline{\mathrm{Bun}}_B$ (Drinfeld compactification)
not smooth $\rightarrow$ not self dual sheaf using above definition!

$\Rightarrow$ modify definition:

$Eis_{\mathcal{E}} = Rp_! q^* \mathcal{E} \otimes IC_{\overline{Bun}_B} \otimes (\bar{\mathbb{Q}}_\ell)_{(\omega_X^{1/2})^{2\rho}}$: self-dual

analog of shift & twist!

3) $G = GL_n$: (1) ~ (2) ... see Eisenstein series

4) Compatibility with classical theory (via fonctions faisceaux)

1) $G = GL_n$: $\overline{Bun}_B^{\text{Laum}}$ = moduli stack of $(\mathcal{L} \supset \mathcal{L}_{n-1} \supset \ldots \supset \mathcal{L}_1)$: $\mathcal{L}_i$ subsheaf of rank i of rank n bundle $\mathcal{L}$.

Projection on $\mathcal{L}$ is <u>proper</u> : space of subsheaves of $\mathcal{L}$ is complete variety, conditions $\mathcal{L}_i \subset \mathcal{L}_{i+1}$ are closed.

$q : \overline{Bun}_B^{\text{Laum}} \to Bun_T$ $q(\mathcal{L} \supset \mathcal{L}_{n-1} \supset \ldots \supset \mathcal{L}_1) = (\mathcal{L}_1, \frac{\det \mathcal{L}_2}{\mathcal{L}_1}, \frac{\det \mathcal{L}_3}{\det \mathcal{L}_2}, \ldots)$

(since embeddings are not necessarily maximal.)

$n > 2$ have outer automorphism of GL_n $g \mapsto (g^t)^{-1}$, induces automorphism of Bun_{GL_n}. <u>But</u> this definition is <u>not</u> stable under this automorphism:

$\mathcal{L} \supset \mathcal{L}_{n-1} \supset \ldots \supset \mathcal{L}_1 \overset{??}{\rightsquigarrow} \mathcal{L}^* \supset \mathcal{L}_{n-1}' \supset \mathcal{L}_{n-2}' \supset \ldots\ldots$

– for vector subbundles just define $\mathcal{L}_k' = \mathcal{L}_{n-k}^{\perp}$. – doesn't generalize to arbitrary subsheaves.

⇝ even for classical groups don't know how to generalize this.

2) $G = GL_n$ $\overline{Bun}_B$: Plücker — V vector space $\supset L$ subspaces, dim k ⇝ Grassmannian. $\Lambda^k L \subset \Lambda^k V$ line, $\in \mathbb{P}\Lambda^k V$, determines L.

$V \supset L_{n-1} \supset L_{n-2} \supset \ldots \Rightarrow \{\Lambda^k L_k \subset \Lambda^k V\}$,

$Fl(V) \hookrightarrow \prod^{n-1} \mathbb{P}(\Lambda^k V)$, image defined by <u>Plücker equations</u>:

$1 \leq i \leq j \leq n$ $f_{ij} : \Lambda^i V \otimes \Lambda^j V \longrightarrow \Lambda^{i-1} V \otimes \Lambda^{j+1} V$

$$\begin{tikzcd} \Lambda^i V \otimes \Lambda^j V \arrow[d] \arrow[r, "f_{ij}"] & \Lambda^{i-1} V \otimes \Lambda^{j+1} V \\ \Lambda^{i-1} V \otimes V \otimes \Lambda^j V \arrow[ur] & \end{tikzcd}$$

$\{l_k \subset \Lambda^k V \text{ lines}\}$ come from flag iff $f_{ij}(l_i \otimes l_j) = 0$ $i \leq j$

$\mathcal{L}$ on X vector bundle of rank n, $\mathcal{L} \supset \mathcal{L}_{n-1} \supset \ldots$ flag of subbundles $\Longrightarrow$

$\mathcal{A}_i = \Lambda^i \mathcal{L}_i \subset \Lambda^i \mathcal{L}$ line subbundles. $\boxed{f_{ij}(\mathcal{A}_i \otimes \mathcal{A}_j) = 0 \quad i \leq j}$
characterize flags of sub-bundles.

$\overline{Bun_B} = \{ (\mathcal{L}, \mathcal{A}_i \subset \Lambda^i \mathcal{L}$ rank 1 subsheaves $\ldots) /$ (on open dense where $\mathcal{A}_i$ are subbundles should come from flag)
$f_{ij}(\mathcal{A}_i \otimes \mathcal{A}_j) = 0 \quad i \leq j$

$Bun_B^{Laum} \ni (\mathcal{L} \supset \mathcal{L}_{n-1} \supset \ldots \supset \mathcal{L}_1) \longmapsto (\mathcal{L}, \mathcal{A}_i = \Lambda^i \mathcal{L}_i)$

$\underline{n=3}$: $\Lambda^2 \mathcal{L} = \mathcal{L}^* \otimes \det \mathcal{L}$

$\overline{Bun_B} = \left\{ \begin{array}{l} (\mathcal{L} \text{ rank } 3, \ \mathcal{A} \subset \mathcal{L} \text{ rank } 1, \ \mathcal{A}' \subset \mathcal{L}^* \text{ rank } 1) \\ \text{s.t. } (\mathcal{A}, \mathcal{A}') = 0 \end{array} \right\}$

Bun_B^{Laum} : fibers $\doteq$ already know maximal possible $\mathcal{L}_2^{max}$ & know generically by orthogonality.
$\mathcal{L}_2^{max} / \mathcal{L}_2$ skyscraper whose lengths we know, but don't know skyscraper itself — get Coh's as fibers...

Since $\det \mathcal{L}_i = \mathcal{A}_i$, can define $q : \overline{Bun_B} \to Bun_T$,
$(\mathcal{L}, \mathcal{A}_i) \longmapsto (\mathcal{A}_1, \mathcal{A}_2 / \mathcal{A}_1, \mathcal{A}_3 / \mathcal{A}_2, \ldots)$

Generalize! $\Lambda^i \mathcal{L}$ are fundamental reps...
If commutant of G $[G,G]$ simply connected this works to define $\overline{Bun_B}$: don't have "the" fund. weights, must extend somehow from semisimple part to whole group $\rightsquigarrow$ some (inessential) arbitrariness.

$\underline{\text{Set theoretically}}$: Each $\mathcal{A}_i \subset \tilde{\mathcal{A}}_i$ maximal - unique.
$\mathcal{A}_i = \tilde{\mathcal{A}}_i(-D_i)$, $D_i \geq 0$ $\quad r$ = semi-simple rank

$(\mathcal{L}, \tilde{\mathcal{A}}_i) \in \overline{Bun_B}$, $D_1 \ldots D_r$ measure degeneracy

- stratification of $\overline{Bun_B}$. $\deg \tilde{\mathcal{A}}_i - \deg \mathcal{A}_i$ constant on components.

Suppose $K \subset H$ algebraic groups.

A K-bundle on $X \longleftrightarrow$ H-bdle $\mathcal{F}$ on X, with K-structure, i.e. an H-morphism $\boxed{\mathcal{F} \to H/K}$

$\longleftrightarrow$ section $X \to (H/K)_{\mathcal{F}}$

$K = B$, $H = G \times T$, $B \hookrightarrow G \times T$. $G \circlearrowright$

$\Rightarrow$ A B-bundle is a triple $(\underset{G\text{-bdle}}{\mathcal{F}_G}, \underset{T\text{-bdle}}{\mathcal{F}_T}, \mathcal{F}_G \times_X \mathcal{F}_T \to (G \times T)/B = G/N)$ $G \times T$-equivariant

<u>Def</u>. A generalized B-bdle is $(\underset{G\text{-bu}}{\mathcal{F}_G}, \underset{T\text{-bu}}{\mathcal{F}_T}, \mathcal{F}_G \times_X \mathcal{F}_T \to \overline{G/N})$ $G \cdot T$ equivariant

$\overline{G/N}$ = affinization of G/N = Spec $\mathrm{Fun}[G/N] \hookleftarrow G/N$.

$G = SL_2$: $G/N = \mathbb{A}^2 \setminus \{0\} \hookrightarrow \overline{G/N} = \mathbb{A}^2$.

- Usually $\overline{G/N}$ not smooth – only when semisimple part is product of rank 1 groups.

$G = GL_n(k)$: G/N = flags $0 \subset V_1 \subset \dots \subset V_{n-1} \subset k^n$, equipped with isomorphism $k \xrightarrow{\sim} \Lambda^i V_i \subset \Lambda^i k^n$; $1 \mapsto x_i \neq 0$: $\{(x_i \in \Lambda^i k^n, \text{Plücker}), x_i \neq 0\}$

$\overline{G/N} = \{(x_i \in \Lambda^i k^n, \text{Plücker})\}$

<u>Equivalence (in GL_n) of definitions</u>:

$\mathcal{F}_G$ = rank n bundle $\mathcal{L}$, $n-1$ line bundles $\mathcal{A}_i$,

x_i ~~...~~ ~~for~~ give $\mathcal{A}_i \to \Lambda^i \mathcal{L}$ satisfying Plücker.

$[G,G]$ not simply connected:

e.g. $G = SO(3) = SL(2)/\pm 1$ $\longrightarrow$ $\mathrm{Bun}_B \subset \overline{\mathrm{Bun}}_B$ <u>not closed</u>!

$\mathcal{L}$ rank 3 orthogonal bundle.

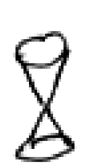
B-structure: isotropic rank 1 $\mathcal{A} \subset \mathcal{L}$ $(\mathcal{A}, \mathcal{A}) = 0$

gen B-structure: " " " subsheaf $\mathcal{A} \subset \mathcal{L}$ "

$\mathcal{A} = \tilde{\mathcal{A}}_{\text{maximal}}(-D)$. e.g. $D = \{x\} \in X$.

$\mathcal{A} = \tilde{\mathcal{A}}(-x)$ is not in closure (generally for deg D odd)

Global explanation: wrong degrees.

Local explanation: look at $D^\times$ around x, trivialize everything
$S^1 \longrightarrow$  cone of isotropic vectors $\setminus \{0\}$ $\subset \mathbb{C}^3$ $\cong \mathbb{A}^2 \setminus \{0\}/\{\pm 1\} = G/N$

$S^1 \to$ cone $\setminus \{0\}$, maps to nontrivial element of fundamental group... (which is $\mathbb{Z}/2$).
So divisor must be even + since all multiplicities are even!

- A point $x \in \overline{G/N} \iff$ collection of linear maps $f_V : V^N \to V$, $V \in Rep_G$, compatible with $\otimes$ and functorial in V.

Note $G/N \times V^N \longrightarrow V$ orbit of N-fixed vectors, extends to $\overline{G/N} \times V^N \longrightarrow V$ (regular functions on G/N extend uniquely!)
$\Rightarrow$ $x \in \overline{G/N}$ gives $V^N \to V$.
Inverse construction: take $V = Fun(G)$,
$f_V : Fun(G/N) \longrightarrow Fun(G) \xrightarrow{\text{eval at } 1} k$
(compatibility with $\otimes \Rightarrow$ this is ring homomorphism!

- $\mathcal{F}_G \underset{x}{\times} \mathcal{F}_T \longrightarrow \overline{G/N} \iff \forall V \in Rep_G$, have
$\varphi_V : (V^N)_{\mathcal{F}_T} \longrightarrow (V)_{\mathcal{F}_G}$ functorial, compatible with $\otimes$

◎ Char $k = 0$: $\{G\text{-mod}\}$ semisimple
(any char irreds $\longleftrightarrow$ highest weights but char p)
(the irreducible are smaller than char 0.

V_λ = irrep of h.w. λ and fixed highest vector, $V^N \cong k_\lambda$
$\varphi_\lambda : (k_\lambda)_{\mathcal{F}_T} \longrightarrow (V_\lambda)_{\mathcal{F}_G}$

Compatibility with $\otimes$: $(V_{\lambda+\mu})_{\mathcal{F}_G} \hookrightarrow (V_\lambda \otimes V_\mu)_{\mathcal{F}_G}$, with $(k_{\lambda+\mu})_{\mathcal{F}_T}$ mapping to both

- works for char p as well - but must replace irred by V_λ - Weyl module (contragredient?)
- generated by one vector, N-invariant, T-weight vector, maximal wrt this property (rest are its quotients)
Char 0: maximal integrable quotient of Verma module

Verma is Vig-module : dual to Fun[G], - this def extends to char p.
$\longleftrightarrow$ use Kempf's B-W-B : $V_\lambda^* = \{ f \in \mathrm{Fun}(G/N),\ f(gt) = t^\lambda f(g) \}$
$= \Gamma(G/B, \mathcal{L}_\lambda)$

3. GL_n equidecomp; Kuznetsov theorem:
$\overline{Bun}_B^{Laum} \xrightarrow[proper]{} \overline{Bun}_B$ is small.

(Small proper morphism from smooth variety has $p_! \mathbb{C}[\cdot] = IC$)

4. Compatibility with classical Eisenstein!
what should stalks be at $\overline{Bun}_B$ given compatibility ?
$\rightarrow$ leads to correct prediction for GL_n (gives smallness etc).
General G : Finkelberg, Mirković, Gaitsgory, Feigin
(with other motivations!)

GL_2 only one L-function
G L-functions labelled by positive coroots
$\Rightarrow$ IC stalks should involve these as well
$\rightarrow$ q-analog of Kostant partition function

Smoothness Bun_K, K general (X smooth, proj...)
- always smooth, $\dim_{\mathcal{F}} Bun_K = -\chi(k_{\mathcal{F}}) =$ (Riemann-Roch)
$\mathcal{F} \in Bun_K$
- follows from deformation theory. Smoothness: extension of deformations to higher order.
Obstruction $\in H^2(X, k_{\mathcal{F}}) = 0$ for curve
$H^1(X, k_{\mathcal{F}}) = T_{\mathcal{F}} Bun_K, \quad H^0(X, k_{\mathcal{F}}) = \mathrm{Lie}(\mathrm{Aut}(\mathcal{F} \in Bun_K))$

$K = B$: adjoint $k_{\mathcal{F}}$ has nontrivial determinant...

- Generalized B-bundles : cover $X = U \cup V$, all "bad" points $\subset V \setminus (U \cap V)$
Try to extend deformation/Artinian rings separately on U, V...
no obstructions on affine schemes, Extensions $\exists$, ! up to non canonical isom $\rightarrow$ problem purely local near bad points.

GL_2 case: $A \hookrightarrow L$ rank two

Locally have $\mathcal{O} \hookrightarrow \mathcal{O} \oplus \mathcal{O}$, i.e. pair of functions, i.e. $V \longrightarrow \overline{G/N} = \mathbb{A}^2$.

want to deform this map $V \to \mathbb{A}^2$.

If $\overline{G/N}$ smooth $\longrightarrow$ answer is yes

$\overline{G/N}$ not smooth - answer is really no.

Same argument works for $\overline{Bun_B}^{Lau}$.

Stalks of IC sheaves for rank $G > 1$: nontrivial here can't be smooth!