V. Drinfeld   Geometric Langlands, Hecke eigensheaves etc.   3/12/97

$X/\mathbb{F}_q$ $g>1$, smooth proj geom irred etc.

E ℓ-adic $SL(2)$ local system on X $\leftrightarrow$ $\rho : \pi_1(X) \to SL(2, \overline{\mathbb{Q}}_\ell)$

$\exists f : Bun_{PGL(2)}(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell$, $f \neq 0$

$\forall x \in X$, $T_x f = Tr\, \rho(Fr_x) f$

Bun_{PGL_2} = rank 2 bundles modulo tensoring with line bundle

$$(T_x f)(\mathcal{L}) = \frac{1}{(-q^{\frac{1}{2}})^{\deg x}} \sum_{\mathcal{L}(-x) \subsetneq \mathcal{L}' \subsetneq \mathcal{L}} f(\mathcal{L}')$$

where $\deg x = [k_x : \mathbb{F}_q]$

- multiple comes from unitary normalization..

GL_2 have two kinds of Hecke operators – T_x' as above

$(T_x^2 f)(\mathcal{L}) = f(\mathcal{L}(\pm x))$ .. for PGL_2 want $T_x^2 f = f$...

Pick ~~a~~ ℓ for all $q^{\frac{1}{2}} \in \overline{\mathbb{Q}}_\ell$ ... not canonical.

PGL_2 we constructed $f(\mathcal{L}) = Tr\,(Fr_y, \mathcal{F})$, $\mathcal{F}$ a perverse sheaf on Bun_{PGL_2} – geometric construction, works for other fields, other characteristics, other cohomology theories..

Write $G = PGL(2)$, ${}^L G = SL(2)$. Choose $x \in X(\mathbb{F}_q)$.

Hecke correspondence:

$$\begin{array}{ccccc} & & \mathcal{H}_x = \{(\mathcal{L}, \mathcal{L}'), \mathcal{L}(-x) \subsetneq \mathcal{L}' \subsetneq \mathcal{L}\} & & \\ & \swarrow^{\pi} & & \searrow^{\varphi} & \\ Bun_G & & & & Bun_G \\ \shortparallel & & & & \shortparallel \\ \{\mathcal{L}\} & & & & \{\mathcal{L}'\} \end{array}$$

$T_x = \frac{1}{(-q^{\frac{1}{2}})^{\deg x}} \pi_* \varphi^*$, $T_x f = Tr\,(Fr_x, E_x) \cdot f$

$T_x : \mathcal{D}(Bun_G) \longrightarrow \mathcal{D}(Bun_G)$, $T_x \mathcal{F} := R\pi_* \varphi^* \mathcal{F}[1](\tfrac{1}{2})$

$\sim$ $T_x \mathcal{F} = E_x \otimes \mathcal{F}$ eigensheaf

$\varphi^*[1]$ – natural to take perverse sheaves to perverse, Tate twist to preserve weight of pure sheaves!

$\mathcal{H} \longrightarrow X$ start with fibers $\mathcal{H}_x$.

$$X \times Bun_G \xleftarrow{\pi} \mathcal{H} \xrightarrow{\varphi} Bun_G$$

$T\mathcal{F} = R\pi_* \varphi^* \mathcal{F}[1](\tfrac{1}{2})$

Eisensheaf: $T\mathcal{F} = E \boxtimes \mathcal{F}$.

In general, role of E played by ${}^LG$-local system $\mathcal{E}$
V^λ = irrep of ${}^LG$, $\Rightarrow$ $\mathcal{E}$-twist $V^\lambda_{\mathcal{E}}$
$\rightsquigarrow T^\lambda \mathcal{F} = V^\lambda_{\mathcal{E}} \boxtimes \mathcal{F}$, Hecke operators T^λ labeled by coweights of G $\longleftrightarrow$ dominant weights of ${}^LG$.

$\mathcal{F}$ in general won't be local system — over char 0, G simply connected $\Rightarrow$ Bun_G is simply connected
Laumon calculated characteristic variety of $\mathcal{F}$... as $\mathcal{D}$-module $\Rightarrow$ we'll pass to deRham world.
X curve/$\mathbb{C}$, ℓ-adic sheaf $\rightsquigarrow$ $\mathcal{D}$-modules
Sing Supp $M_E \subset T^* Bun_G$ Lagrangian conic
M_E $\mathcal{D}$-module, corresponding to $\mathcal{F}$.
Laumon (1987) S.S. M_E = the zero fiber of Hitchin's fibration.
(in particular is indep. of E)
$F \in Bun_G$, $T_F Bun_G = H^1(X, \mathfrak{g}_F)$.
$T^*_F Bun_G = H^0(X, \mathfrak{g}_F \otimes \omega_X)$. $\qquad$ $\det : sl_2 \longrightarrow \mathbb{C}$
$\det \eta \in H^0(X, \omega_X^{\otimes 2})$
$(F, \eta) \longmapsto \det \eta$ is Hitchin's map in this case.

S.S. M is a cycle have component multiplicities — and these agree with those of Hitchin zero fiber.
$p^{-1}(0) = \{(F, \eta) \mid \eta \text{ is nilpotent}\}$
Multiplicity of $\{\eta = 0\}$ in $p^{-1}(0)$ for PGL_2 is 2^{3g-3}
$3g-3 = \dim Bun_{PGL_2} = \dim H^0(X, \omega_X^{\otimes 2})$.

Hitchin: p is Lagrangian fibration (all fibers Lagrangian)
$\forall$ functions φ, ψ on Hitchin base $\{p^*\varphi, p^*\psi\} = 0$

<u>General Strategy</u>
$A_0 = \{\text{Functions on } H^0(X, \omega_X^{\otimes 2})\} \simeq \mathbb{C}[u_1, \ldots u_N]$ (noncanonical)
$N = 3g-3$
Hitchin map $\Rightarrow$ $p^* : A_0 \hookrightarrow \{\text{Functions on } T^* Bun_G\}$

(graded embedding - p is $\mathbb{C}^*$ equivariant..)
Naive idea: construct commutative $A \subset H^0(Bun_G, \mathcal{D})$
s.t. $gr A = A_0$ (wrt standard filtration.
In fact (Beilinson - Kazhdan) $H^0(Bun_G, \mathcal{D}) = \mathbb{C} \times \mathbb{C}$.

$\mathcal{D}'$ = sheaf of diffops on $\omega^{\frac{1}{2}}_{Bun_G}$
1. Construction of commutative $A \subset H^0(Bun_G, \mathcal{D}')$, $gr A = A_0'$.
2. $m \subset A$ max ideal, $M' := \mathcal{D}'/\mathcal{D}'m$
- we have $L_1 \mapsto \lambda_1, \ldots L_N \mapsto \lambda_n$ for generators $\{L_i\}$ of A
$\Rightarrow$ system of equations $L_i \psi = \lambda_i \psi$.
M' holonomic, s.s. $M' = p^{-1}(0)$
- holonomic since $ch\, M' = p^{-1}(0)$ by looking at principal symbols.
3. $M = \omega^{-\frac{1}{2}}_{Bun_G} \otimes_{\mathcal{O}} M'$ is a $\mathcal{D}$-module
(here need $\omega^{\frac{1}{2}}$ as a <u>line</u> bundle, not just TDO).
(assume for now $G = SL_2 \Rightarrow \omega^{\frac{1}{2}}$ is unique)

Fact $Spec\, A = \{$ projective connections on $X \}$
- proj conn = Sturm-Liouville operator $L: \omega_x^{-\frac{1}{2}} \to \omega_x^{3/2}$
self adjoint operator of order 2, principal symbol 1...
$\leftrightarrow$ symbol of $L: A \to B$ taking $A \otimes \omega_x^{2} \to B$
Adjoint takes $\omega \otimes B^{-} \to \omega \otimes A \Rightarrow A \otimes B = \omega_x$,
$B \cong A \otimes \omega_x^2$. Well defined up to twisting by
A s.t. $A^{\otimes 2} = \mathcal{O}_x$.
Twisting as such tensors and rep $\pi_1 \to SL_2$ by
rep to $\{\pm 1\} \Rightarrow$ well defined PSL_2 connection.

Drinfeld II - Hitchin's fibration & opers

G, X genus $g>1$.

$p: T^*Bun_G \longrightarrow Hitch_{\mathfrak{g}}(X)$

$\mathcal{F} \in Bun_G$, $T_{\mathcal{F}} Bun_G = H^1(X, \mathfrak{g}_{\mathcal{F}})$,

$T^*_{\mathcal{F}} Bun_G = H^0(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X) \ni \eta$

$(\mathcal{F}, \eta) \in T^*Bun_G$, apply invariant polynomials:

$Inv(\mathfrak{g}^*) = \mathbb{C}[f_1, \dots f_r]$, $\deg f_i = k_i$ & r = rank $\mathfrak{g}$.

$f_i(\eta) \in H^0(X, \omega_X^{\otimes k_i})$

$\Rightarrow$ $p: T^*Bun_G \longrightarrow \bigoplus_i H^0(X, \omega_X^{\otimes k_i}) = Hitch_{\mathfrak{g}}(X)$

$(\mathcal{F}, \eta) \longmapsto (f_1(\eta), \dots, f_r(\eta))$

$\dim Bun_G = \dim H^1(X, \mathfrak{g}_{\mathcal{F}}) - \dim H^0(X, \mathfrak{g}_{\mathcal{F}})$ ← # of infi. automorphisms! Bun_G as stack

$= (\dim G)\cdot(g-1)$

$\dim Hitch_{\mathfrak{g}}(X) = \sum_i \dim H^0(X, \omega_X^{\otimes k_i}) = \sum_i (g-1)(2k_i - 1)$

but $\sum_i (2k_i - 1) = \dim \mathfrak{g}$

$\Rightarrow$ dimensions equal.

More invariantly, $Hitch_{\mathfrak{g}}(X) = Mor_{graded}(Inv(\mathfrak{g}^*), \bigoplus_j H^0(X, \omega_X^{\otimes j}))$

In general no natural structure of vector space on $Hitch_{\mathfrak{g}}(X)$ - involves choice of generators of $Inv(\mathfrak{g}^*)$ - naturally it is an affine algebraic variety $\cong \mathbb{A}^N$, $N = \dim Bun_G$.

It does have natural $\mathbb{G}_m$ action -

λ acts on $\omega_X^{\otimes k_i}$ as λ^{k_i} ..

Introduce $C = Spec\, Inv(\mathfrak{g}^*) = W \backslash \mathfrak{h}^*$ (naive quotient)

Then $Hitch_{\mathfrak{g}}(X) = \Gamma(X, C_{\omega_X})$.

C is variety with $\mathbb{G}_m$ action,

ω_X is $\mathbb{G}_m$-torsor $\longrightarrow$ take twist C_{ω_X}.

In this description, Hitchin map $p: T^*Bun_G \to \Gamma(X, C_{\omega_X})$ is twist by $\mathcal{F}, \omega$ of map $Inv: \mathfrak{g}^* \to C$

$T^*_{\mathcal{F}} Bun_G = \Gamma(\mathfrak{g}_{\mathcal{F}}^* \otimes \omega) \longrightarrow \Gamma(X, C_{\omega_X})$ . . .

Opers

G reductive

Y a smooth curve (not nec. complete, maybe just formal..)

Take $G = GL(n)$.

Def. A $GL(n)$-oper on Y is $(\underset{\text{rk } n}{\mathcal{L}}, \nabla : \mathcal{L} \to \mathcal{L} \otimes \omega_x,$

$0 \subset \mathcal{L}_1 \subset \dots \subset \mathcal{L}_{n-1} \subset \mathcal{L}$ complete flag of subbundles),

s.t. 1) $\nabla(\mathcal{L}_i) \subset \mathcal{L}_{i+1} \otimes \omega_x$

2) The $\mathcal{O}$-linear map $\nabla : \mathcal{L}_i / \mathcal{L}_{i-1} \to (\mathcal{L}_{i+1}/\mathcal{L}_i) \otimes \omega_x$ is iso.

Locally can write $\nabla_{\frac{d}{dz}} = \frac{d}{dz} + q(z)$

$q(z) = \begin{pmatrix} * & & * \\ x & \ddots & \\ 0 & x & \ddots \end{pmatrix}$ with x's $\neq 0$.

A G-oper is a G-bundle $\mathcal{F}$ with connection ∇, B-structure on $\mathcal{F}$

$q(z)$ in $\mathfrak{b}$ except for negative simple roots, which shouldn't vanish.

A $\mathfrak{g}$-oper is a G_{ad}-oper, adjoint group for $\mathfrak{g}$ semisimple.

Suppose Y complete, $g > 1$.

a. For given $(\mathcal{F}, \nabla)$ $\exists$ at most one flag on $\mathcal{F}$ st. $\mathcal{F}, \nabla$ becomes an oper.

b. $(\mathcal{F}, \nabla)$ is "irreducible" (can't be reduced, as loc. system, to a parabolic)

a. Recall $\nabla : \mathcal{L}_i / \mathcal{L}_{i-1} \to \mathcal{L}_{i+1}/\mathcal{L}_i \otimes \omega_x$

so degree gets successively smaller : $\deg(\mathcal{L}_i/\mathcal{L}_{i-1}) > \deg(\mathcal{L}_{i+1}/\mathcal{L}_i)$

$\Rightarrow$ this is the Harder–Narasimhan flag –

pick line subbundle of highest degree ...

b. is exercise.

$\Rightarrow$ $Op_G(X) \subset LocSys_G(X)$, as Lagrangian subspace...

Classical opers

$\nabla = -\frac{d}{dz} + q(z)$ $\mathfrak{gl}(n)$ oper

$\Rightarrow \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \quad \nabla e_1 = \begin{pmatrix} * \\ \neq 0 \\ 0 \\ \vdots \end{pmatrix} \quad \ldots$

so $e_1, \nabla e_1, \ldots \nabla^{n-1} e_1$ is a basis of sections locally

$$\nabla^n(e_1) + \sum_0^{n-1} a_i(z)\nabla^i(e_1) = 0 \Rightarrow L = \left(\frac{d}{dz}\right)^n + \sum_0^{n-1} a_i(z)\left(\frac{d}{dz}\right)^i$$

$\Rightarrow$ diff eq $Le_1 = 0$.

Alternatively, $\left(\frac{d}{dz} + q(z)\right)\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} = 0$

express ψ_i in terms of $\psi_n \Rightarrow$ adjoint operator to L.

$\{GL(n)\text{-opers}\} \longleftrightarrow \{\text{Diffops } A \longrightarrow A \otimes \omega_X^n \text{ order } n, \text{ principal symb. } 1\}$

A some line bundle.

Symbol of n^{th} order operator $L: A \longrightarrow B$ is map $A \otimes \omega^n \longrightarrow B$.

In terms of oper flag $0 \subset \mathcal{L}_1 \subset \ldots \subset \mathcal{L}$, $A = \mathcal{L}_1^{\otimes -1}$

$\{Sp(n)\text{-opers}\} \longleftrightarrow \{\text{selfadjoint diffops order } n, \ L: A \to A \otimes \omega_X^n \text{ prin. symb. } 1\}$

Note $L: A \longrightarrow B$ adjoint operator takes $\omega_X \otimes B^{-1} \longrightarrow \omega_X \otimes A^{-1}$

$\Rightarrow$ need $A^{\otimes 2} = \omega_X^{\otimes(1-n)}$

$\{PSL_2\text{-opers} = \mathfrak{sl}_2\text{-opers}\}$ - must allow twist by μ_2 torsors:
identify different choices of $\omega^{\frac{1}{2}}$.

$Op_{\mathfrak{g}}(X)$ & $Hitch_{\mathfrak{g}}(X)$

Write $Hitch_{\mathfrak{g}}(X) = \mathrm{Spec}\, A_0$.

In fact $Op_{\mathfrak{g}}(X) = \mathrm{Spec}\, A$, A has natural filtration with $\mathrm{gr}\, A = A_0$.

Hitchin's map p gives embedding $A_0 \hookrightarrow$ Functions on $T^*Bun_{^LG}$, which we want to quantize .. we'll identify spec $\mathcal{D}_{crit}(Bun_G)$ with opers . . .

Introduce ε-opers, $\varepsilon \in \mathbb{C}$. $\varepsilon = 1$ obtain usual opers.
$\varepsilon = 0$ obtain $Hitch_{\mathfrak{g}}(X)$

– flat family of schemes $\Rightarrow$ $\begin{array}{c} Op_{\mathfrak{g}}(X) \\ \pi \downarrow \\ \mathrm{Spec}\, \mathbb{C}[\varepsilon] \end{array}$

Action of $\mathbb{G}_m$ on base $\mathbb{A}^1_{\varepsilon}$ to $\underline{Op}_{\mathfrak{g}}(X)$

$\underline{Op}_{\mathfrak{g}}(X) = \operatorname{Spec} \underline{A}$, flat graded algebra over $\mathbb{C}[\varepsilon]$.

$\underline{A} = \bigoplus_{k \geq 0}^{\infty} A_k \qquad \underline{A}_0 \xhookrightarrow{\varepsilon} \underline{A}_1 \xhookrightarrow{\varepsilon} \underline{A}_2 \longrightarrow \cdots$

$A = \underline{A}/(\varepsilon - 1) = \varinjlim A_k$

$A_0 = \underline{A}/\varepsilon = \bigoplus \underline{A_k / A_{k-1}}$ - filtration on A with associated graded A_0.

ε-connection: $\mathcal{L}$ $\mathcal{O}_X$-module, an ε-connection on $\mathcal{L}$ is
$\nabla : \mathcal{L} \to \mathcal{L} \otimes \omega_X$, $\mathbb{C}$-linear, $\nabla(f l) = f \nabla(l) + \varepsilon\, l \otimes df$,
f function, l section.

∇ ε-connection $\Rightarrow$ $c\nabla$ is $c\varepsilon$-connection.

What is an ε-oper for $\varepsilon = 0$?
$\to$ A B-bundle $\mathcal{F}$ with $q \in H^0(X, \mathfrak{g}_{\mathcal{F}} \otimes \omega)$,
locally q is in $\begin{pmatrix} * & * & * \\ * & \ddots & \\ 0 & \ddots & * \end{pmatrix} =: \mathfrak{g}^*_{-1}$

Gauge transformations $\longrightarrow$ B conjugation (at $\varepsilon = 0$).

Fix $x \in X$, trivialize $T^*_x X$

$\underline{B}$ acts on $\mathfrak{g}^*_{-1}$ by conjugation

<u>Kostant</u> The B-action on $\mathfrak{g}^*_{-1}$ is free & induces an isomorphism $B \backslash \mathfrak{g}^*_{-1} \xrightarrow{\sim} W \backslash \mathfrak{h}$.

Thus an ε-oper for $\varepsilon = 0$ is an element of $\Gamma(X, \mathcal{C}_{\omega_X})$
$\mathcal{C}_{\omega_X} = W \backslash \mathfrak{h}$ which is same as before - we dealt with dual Lie algebra & took $W \backslash \mathfrak{h}^*$ ... ~~Same~~

Fix a principal $\mathfrak{sl}_2$ triple $\{h, e, f\}$ $\qquad e \in \begin{pmatrix} 0 & \neq 0 & \\ & \neq 0 \ddots & 0 \\ 0 & & \end{pmatrix}$ etc.

Consider $V = \ker \operatorname{ad} e \subset \mathfrak{g}$, $\dim V = r$.

$f + V \subset \mathfrak{g}^*_{-1} \quad \& \quad f + V \xrightarrow[\text{"canonical"}]{} B \backslash \mathfrak{g}^*_{-1}$ isomorphism

$\Rightarrow$ Canonical form for opers:

$$\nabla_{\frac{d}{dz}} = \frac{d}{dz} + f + q(z), \quad q(z) \in V \qquad \text{(locally)}$$

$H|_{sl_2 \text{ principal}} = \begin{pmatrix} \lambda_1 & & 0 \\ 0 & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \lambda_i / \lambda_{i+1} = \lambda$

- using this can correct transformation properties:

$\frac{d}{dz} = \varphi \frac{d}{d\zeta} \Rightarrow \varphi f$ appears..

Write $\nabla_{\frac{d}{dz}} = \frac{d}{dz} + u(z) f - v(z) h + q(z)$, $q(z) \in V$

modulo $\tilde{\nabla} = b(z)^{-1} \nabla b(z)$, $b(z) \in B_{PSL(2)}$, Lie $B_{sl(2)} = \mathbb{C}e \oplus \mathbb{C}h$

- this form is stable under local change of coordinates

V is h-graded, $V = \bigoplus_m V_m$, $V_m = \{ v / [h\, v] = 2m v \}$

$V_m \neq 0 \Leftrightarrow m+1 \in \{k_1, \dots k_r\}$ basis of invariant polys.

$v_1 \dots v_r$ a basis of V, $v_i \in V_{k_i - 1}$, $v_1 = e$.

$\nabla_{\frac{d}{dz}} = \frac{d}{dz} + f + \sum \varphi_i(z) v_i$

Exercise φ_1 is projective connection rest are differentials of corresponding degree: $\varphi_i(z) dz^{k_i}$ is invariant $i > 1$,

$(\frac{d}{dz})^2 - \varphi_1(z)$ is invariant operator $\omega_X^{-\frac{1}{2}} \to \omega_X^{\frac{3}{2}}$.

$\Rightarrow Op_{\mathfrak{g}}(X) = \{ \text{proj. conn on } X \} \times \bigoplus_{i=2}^{r} H^0(X, \omega_X^{\otimes k_i})$

$\text{Hitchin}_{\mathfrak{g}}(X) = \bigoplus_{i=1}^{r} H^0(X, \omega_X^{\otimes k_i})$

- "canonical" form gives a vector space structure on W/h.

Op_{sl_2} is a torsor over $H^0(X, \omega_X^{\otimes 2})$

$Op_{\mathfrak{g}}$ is the induced torsor with respect to inclusion

$H^0(X, \omega^{\otimes 2}) \hookrightarrow W = \bigoplus_m V_m \otimes H^0(X, \omega_X^{\otimes(m+1)})$

given by $e \in V_1$:

construct map $Op_{sl_2} \times_{H^0(X, \omega^2)} W \xrightarrow{\sim} Op_{\mathfrak{g}}(X)$

sl_2-oper $(\mathcal{F}_0, \nabla_0)$ with Borel $\mathcal{B}$ induces

G_{ad} bundle $\mathcal{F}, \nabla$ by pushforward,

change induced connection by twists of elements of V.

$\mathfrak{g}_{\mathcal{F}} \supset \mathfrak{g}_{\mathcal{F}_0} \supset V_{\mathcal{F}_0}$, $(V_m)_{\mathcal{F}_0} \to V_m \otimes \omega_X^{\otimes m}$. $\mathcal{B}_0 \to H_0 = \mathbb{G}_m$

- can add sections $w \in W$ to our connection.

V. Drinfeld - Quantizing the Hitchin System 3/28

$p: T^*\mathrm{Bun}_G \to \mathrm{Hitch}_g(X) = \mathrm{Spec}(A_0)$

$\Rightarrow \quad \varphi_0 : A_0 \to \mathrm{Fun}(T^*\mathrm{Bun}_G)$

Hitchin's commutativity theorem: $\varphi_0(A_0)$ is Poisson-commutative.

We have filtered commutative A, $\mathrm{gr}\, A = A_0$,

$\mathrm{Spec}\, A = \mathrm{Op}_{{}^Lg}(X)$

Problem: construct filtered morphism $\varphi: A \to H^0(\mathrm{Bun}_G, D')$ with $\mathrm{gr}\, \varphi = \varphi_0$

$x \in X$, O_x = completed local ring, $\underline{x} = \mathrm{Spec}\, O_x$.

$\mathrm{Bun}_{G,\underline{x}} = \{$ G-bundles on X trivialized over $\underline{x}$ $\}$

- ~~a~~ pro-variety, scheme of infinite type.

$G(O_x)$ acts by change of trivialization... consider $G(O_x)$ as group...

For Y any scheme denote Y_{O_x} scheme representing

$Y_{O_x}(R) := Y(R \hat{\otimes} O_x)$ .. $O_x \simeq \mathbb{C}[[t]]$ and

$R \hat{\otimes} \mathbb{C}[[t]] = R[[t]]$.

e.g. Y affine line get inf-dim affine space $\simeq \mathrm{Spec}\, \mathbb{C}[u_1, u_2, \ldots]$

$\Rightarrow$ group G_{O_x}, get $\mathrm{Bun}_G = G_{O_x} \backslash \mathrm{Bun}_{G,\underline{x}}$

If K_x = fraction field of O_x, then

$G(K_x)$ acts on $\mathrm{Bun}_{G,\underline{x}}$ ..

again for any Y introduce functor $Y_{K_x}(R) := Y(R \hat{\otimes} K_x)$,

where $R \hat{\otimes} \mathbb{C}((t)) = R((t))$

— must assume Y affine to get something reasonable..

functor is ind-representable, by ind scheme which is inductive limit of closed embeddings of schemes..

$\Rightarrow$ group scheme G_{K_x}

(we want G_{O_x}, G_{K_x} not just as sets, but as schemes..)

G_{K_x} acts on $\mathrm{Bun}_{G,\underline{x}}$:

Glueing lemma: consider groupoid

$\{$G-bundles on $X\}$ $\longrightarrow$ $\{$G bundle on $X \setminus x$, G-bun on $\underline{x}$, glueing over $\mathrm{Spec}\, K_x$ $\}$

This functor is in fact an equivalence, & true for S-families of G-bundles (S affine..)

$\{\text{point in } Bun_{G,x}\} = \{G\text{-bun. on } X\setminus x,\ \text{trivialized over } \operatorname{Spec} K_x\}$

· Beauville-Laszlo CRAS 320 (1995) No. 3 335-370

Set $M = Bun_{G,x}$, $H = G_{K_x}$, $K = G_{O_x}$

$H \supset K$, H acts on M. $K\backslash M$ exists as an alg. stack...

assume for convenience everything is fin dim, free actions etc.

How to construct differential operators on the quotient?

~ suppose $K\backslash M$ affine scheme, $M = \operatorname{Spec} A$

K connected, $k = \operatorname{Lie} K$. $\Rightarrow$ $K\backslash M = \operatorname{Spec} A^k$

$h = \operatorname{Lie} H$. h acts on A, Uh acts on A

V an h-module... what acts on V^k? $\Rightarrow$

$\operatorname{Hecke}(h,k)$: $Uh \supset I = (Uh)\cdot k$,

normalizer of I $N(I) = \{a \in Uh,\ Ia \subset I\}$

$= \{a \mid ka \in I\} = \{a \mid [k,a] \subset I\}$

$\operatorname{Hecke}(h,k) = N(I)/I$: $v \in V^k, a \in N(I) \Rightarrow Iav = Iv = 0 \ldots$

Vacuum module: $Vac = (Uh)/(Uh)\cdot k$

$|0\rangle$ = image of $1 \in Uh$.

$k|0\rangle = 0$. ..

$V^k = \operatorname{Hom}_h(Vac, V)$, so our Hecke algebra is

$(\operatorname{End} Vac)^o$, which acts on V^k ... more precisely

$\operatorname{Hecke}(h,k) \longrightarrow (\operatorname{End} Vac)^o$, & is bijective.

$\operatorname{End} Vac = Vac^k$: endo determined by image of $|0\rangle$.

$N(I)/I \subset U(h)/I = Vac \ldots$

$\Rightarrow$. $\operatorname{Hecke}(h,k) \longrightarrow H^0(K\backslash M, \mathcal{D})$.

Classical analog - construct symbols on $K\backslash M$.

$\operatorname{Hecke}_{cl}(h,k) \longrightarrow \operatorname{Fun}(T^*(K\backslash M))$, Poisson

morphism..

Sym $\mathfrak{h}$ has a Poisson bracket (K-K bracket).

$I_{cl} = (\mathrm{Sym}\,\mathfrak{h})\mathfrak{k}$. Take Poisson normalizer

$N(I_{cl}) = \{ a \in \mathrm{Sym}\,\mathfrak{h} : \{a, I_{cl}\} \subset I_{cl} \}$

$$\Rightarrow \quad \mathrm{Hecke}_{cl}(\mathfrak{h},\mathfrak{k}) := N(I_{cl})/I_{cl}$$
$$= (\mathrm{Sym}\,\mathfrak{h} / (\mathrm{Sym}\,\mathfrak{h})\mathfrak{k})^{K}$$
$$= \mathrm{Sym}(\mathfrak{h}/\mathfrak{k})^{K}$$

Moment map $\mu : T^*M \longrightarrow \mathfrak{k}^*$

$T^*(K\backslash M) = K\backslash \mu^{-1}(0)$

$\mathrm{Fun}(T^*(K\backslash M)) = (\mathrm{Fun}(T^*M)/\mathrm{Fun}(T^*M)\cdot\mathfrak{k})^{K}$

$= \mathrm{Hecke}_{cl}(\mathrm{Fun}\,T^*M, \mathfrak{k})$

(note - definition of Hecke_{cl} works for any Poisson algebra..)

- this gives the Poisson structure on $T^*(K\backslash M)$.

$\mathrm{Sym}\,\mathfrak{h} \longrightarrow \mathrm{Fun}(T^*M)$, get map

$\mathrm{Hecke}_{cl}(\mathrm{Sym}\,\mathfrak{h}, \mathfrak{k}) \longrightarrow \mathrm{Hecke}_{cl}(\mathrm{Fun}\,T^*M, \mathfrak{k}) = \mathrm{Fun}(T^*(K\backslash M))$

Classical & quantum constructions are compatible:

Hecke$(\mathfrak{h},\mathfrak{k})$ has filtration from that of $U\mathfrak{h}$ or of Vac, filtered map $\mathrm{Hecke}(\mathfrak{h},\mathfrak{k}) \longrightarrow \mathrm{Diff}(K\backslash M)$

$$\begin{tikzcd}
\mathrm{gr}\,\mathrm{Hecke}(\mathfrak{h},\mathfrak{k}) \arrow[r] & \mathrm{gr}\,\mathrm{Diff}(K\backslash M) \\
\mathrm{Hecke}_{cl}(\mathfrak{h},\mathfrak{k}) \arrow[r] \arrow[u] & \mathrm{Fun}(T^*(K\backslash M)) \arrow[u]
\end{tikzcd}$$

not in general an isomorphism

$$\begin{tikzcd}
T^*M \arrow[r] \arrow[dr, "\mu"'] & \mathfrak{h}^* \arrow[d] \\
 & \mathfrak{k}^*
\end{tikzcd}
\qquad
\begin{tikzcd}
T^*(K\backslash M) = K\backslash \mu^{-1}(0) \arrow[d] & \\
K\backslash(\mathfrak{h}/\mathfrak{k})^* \arrow[r] & \mathrm{Spec}\,\mathrm{Hecke}_{cl}(\mathfrak{h},\mathfrak{k})
\end{tikzcd}$$

Return to setting $M = \mathrm{Bun}_{G,X}$ $\quad H = G_{K_x}$ $\quad K = G_{\mathcal{O}_x}$.

Suppose D finite subscheme of $X \Rightarrow \mathrm{Bun}_{G,D}$ - bundles trivialized on D

$$T_{\mathcal{F}}\,\mathrm{Bun}_{G,D} = H^1(X, \mathfrak{g}_{\mathcal{F}}(-D))$$

$$T^*_{\mathcal{F}}\,\mathrm{Bun}_{G,D} = H^0(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X(D))$$

Set $D = nx$, $n \to \infty$.

$$T^* \mathrm{Bun}_{G,\bar{x}} = \{ \mathcal{F}, \text{ trivialization over } \bar{x},\ \eta \in H^0(X \setminus x, \mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X) \}$$

moment map $\downarrow$ $\quad \swarrow \ \eta|_{\mathrm{Spec}\, K_x}$

$$(\mathfrak{g} \otimes K_x)^* = \mathfrak{g}^* \otimes \omega_{K_x}$$

$$T^* \mathrm{Bun}_G \to G(O_x) \backslash (\mathfrak{g} \otimes K_x / \mathfrak{g} \otimes O_x)^* = G(O_x) \backslash (\mathfrak{g}^* \otimes \omega_{O_x}) \to \mathrm{Spec}\, B$$

$$B = \mathrm{Hecke}_{cl}(\mathfrak{g} \otimes K_x, \mathfrak{g} \otimes O_x) =$$

$= \{ G(O_x)\text{-invariant functions on } \mathfrak{g}^* \otimes \omega_{O_x} \}$

Pick $f_1 \dots f_r \in \mathrm{Inv}(\mathfrak{g}^*)$, $\deg f_i = k_i$

$\eta \in \mathfrak{g}^* \otimes \omega_{O_x} \longmapsto f_i(\eta) \in \omega_{O_x}^{k_i}$

Ignore $\eta \mapsto n$th coefficient of $f_i(\eta)$

Writing $\mathrm{Hitch}_{\mathfrak{g}}(O_x) = \mathrm{Mor}_{\text{graded}}(\mathrm{Inv}(\mathfrak{g}^*), \hat{\oplus}\, \omega_{O_x}^{n})$

$= \mathrm{Spec}\, B$.

$\Rightarrow$ map $\quad T^* \mathrm{Bun}_G \to \mathrm{Hitch}_{\mathfrak{g}}(O_x)$

usual H. map $\leftarrow \quad \to \mathrm{Hitch}_{\mathfrak{g}}(X) \nearrow$

Thus $\mathrm{Hecke}_{cl}(\mathfrak{g} \otimes K_x, \mathfrak{g} \otimes O_x) \to \mathrm{Fun}(T^* \mathrm{Bun}_G)$

has a <u>big</u> kernel.

<u>How to prove a Hecke algebra is commutative</u> ?

$\exists$ obvious map $\quad$ Center of $U\mathfrak{h} = Z \to \mathrm{Hecke}(\mathfrak{h}, k)$ ...

If it's surjective then Hecke is commutative...

Same in classical case (in Poisson setting)

Our functions in map to $\mathrm{Spec}\, B$, come from invariant functions on $\mathfrak{g}^* \otimes \omega_{K_x} \Rightarrow$ it's commutative.

<u>Exercise</u> (!) $\quad \mathfrak{g}$ <u>any</u> Lie algebra (fin. dim.)

$\to$ ~~map~~ $\quad Z = \{ (\mathfrak{g} \otimes K_x)\text{-invariant fns on } \mathfrak{g}^* \otimes \omega_x \}$

$\downarrow \qquad\qquad \downarrow$

$\mathrm{Hecke}_{cl}(\mathfrak{g} \otimes K_x, \mathfrak{g} \otimes O_x) = \{ (\mathfrak{g} \otimes O_x)\text{-invariant functions on } \mathfrak{g}^* \otimes \omega_{O_x}) \}$

is onto ... In fact can replace $\mathfrak{g}^*$ by any $\mathfrak{g}$-module

V. Drinfeld - Quantization of Hitchin continued. 4/2

Recall $K \subset H$, H acts on M $\Rightarrow$

1. $\mathrm{Hecke}(\mathfrak{h}, K) \longrightarrow H^0(K\backslash M, \mathcal{D})$
2. $\mathrm{Hecke}_{cl}(\mathfrak{h}, K) \longrightarrow \mathrm{Fun}(T^*(K\backslash M))$

In setting of $\mathrm{Bun}_{G,x}$ & G_{K_x} action, classical construction gives Hitchin system... but above quantum construction is $\mathrm{Hecke}(\mathfrak{g}\otimes K_x, \mathfrak{g}\otimes \mathcal{O}_x) = \mathbb{C} \longrightarrow \mathbb{C}^{\pi_0(G)}$ ... trivial...

$\Rightarrow$ need twisted differential operators:

$\mathcal{D}'$ = diffops on $\omega^{\frac{1}{2}}_{\mathrm{Bun}_G}$.

– $G(K_x)$ doesn't act on lift of bundle to $\mathrm{Bun}_{G,x}$, rather need a central extension..

Fact The action of $\mathfrak{g}\otimes K_x$ on $\mathrm{Bun}_{G,x}$ lifts canonically to action of $\widetilde{\mathfrak{g}\otimes K_x}$ on the pullback of $\omega^{\frac{1}{2}}_{\mathrm{Bun}_G}$, with critical level.

K-M algebra: $0 \to \overset{\mathbb{1}}{\mathbb{C}} \to \widetilde{\mathfrak{g}\otimes K_x} \to \mathfrak{g}\otimes K_x \to 0$

corresponds to cocycle $\langle u, v\rangle = \mathrm{Res}(du, v)$

Definition The critical scalar product on $\mathfrak{g} := -\frac{1}{2}\cdot$ Killing form

(this is the one we'll use to define $\widetilde{\mathfrak{g}\otimes K_x}$.)

$\rightarrow$ Critical level $\leftrightarrow$ level 1, $\mathbb{1}$ acts by 1.

$(\omega_{\mathrm{Bun}_G})_F = \det R\Gamma(X, \mathfrak{g}_F)$.

Given G-module V, $\Rightarrow$ line bundle $L_F = \det R\Gamma(X, V_F)$

On the pullback of L to $\mathrm{Bun}_{G,x}$ acts the KM algebra with level corresp to scalar product defined by the representation V (times -1)

$(u,v)_V = \mathrm{Tr}(\rho_V(u)\rho_V(v)) \Rightarrow$ level -1.

Quantization: need twisted version of quantum Hecke algebra.. it can be defined as dual to $\mathrm{End}(\mathrm{Vac})$ $\Longrightarrow$ use twisted Vacuum module over $\widetilde{\mathfrak{g}\otimes K_x}$

Vac: $|0\rangle$ generator $\mathfrak{g}\otimes\mathcal{O}_x|0\rangle = 0$,

$\mathbb{1}|0\rangle = |0\rangle$.

$\mathrm{Hecke}'(\mathfrak{g}\otimes K_x, \mathfrak{g}\otimes \mathcal{O}_x) = (\mathrm{End}_{\mathfrak{g}\otimes K_x} \mathrm{Vac})^{opp} =: \mathfrak{z}_{\mathfrak{g}}(\mathcal{O}_x) = \mathfrak{z}(\mathcal{O}_x) = \mathfrak{z}_x$

$\downarrow$

$H^0(\mathrm{Bun}_G, \mathcal{D}')$. Hecke' is big enough

In our situation, stacks are locally of form $G\backslash M$...
want to consider G-invariant diffops on M as diffops on quotient.

$\mathcal{O} := \mathbb{C}[[t]]$ for now.

<u>Feigin-Frenkel</u> 1.) $\mathfrak{z}_{\mathfrak{g}}(\mathcal{O})$ is commutative (true in more general context - not just critical level, $\mathfrak{g}$ semisimple etc. ...)

Moreover the center $Z \subset \bar{U}'(\mathfrak{g}\otimes K)$ maps <u>onto</u> $\mathfrak{z}_{\mathfrak{g}}(\mathcal{O})$.
[Here $U'(\mathfrak{g}\otimes K) := (U\widehat{\mathfrak{g}\otimes K})/(\mathbb{1}-1)$. Topologize this: neighs of 0 formed by left ideals generated by open subalgebras in $\mathfrak{g}\otimes\mathcal{O}$ - so that <u>discrete modules</u> are continuous in this topology - take completion.]

2.) (only $\mathfrak{g}$ semisimple, critical level) $\exists$ <u>canonical</u> isomorphism
$\varphi : \mathfrak{z}(\mathcal{O}) \xrightarrow{\sim} A(\mathcal{O})$ where $A(\mathcal{O}) = \mathrm{Fun}(Op_{{}^L\mathfrak{g}}(\mathcal{O}))$.
(These are discrete rings - not topological $\mathfrak{z}$. Vac is discrete space).

2'.) $\exists$ canonical isom $\phi : Z \xrightarrow{\sim} \mathrm{Fun}(Op_{{}^L\mathfrak{g}}(K))$
(these <u>are</u> topological objects.) $\quad \mathfrak{z}(\mathcal{O}) \xrightarrow{\sim} A(\mathcal{O})$

(Some) Properties:

1. φ is compatible with filtration: on Hecke comes from that on Vac or U', on opers have filtration with graded Fun(Hitch$_{{}^L\mathfrak{g}}(\mathcal{O})$.)
2. $gr\, \varphi = id$, under identification $gr\, \mathfrak{z}(\mathcal{O}) \xrightarrow{\sim} gr A(\mathcal{O})$
$= \mathrm{Fun}(\mathrm{Hitch}_{\mathfrak{g}}(\mathcal{O})) = \mathrm{Hecke}_{cl}(\mathfrak{g}\otimes K, \mathfrak{g}\otimes\mathcal{O}_x) \to$ in fact iso.
- i.e. our Hecke algebra is as big as possible.
3. φ is <u>Aut</u> $\mathcal{O}$ equivariant.

<u>Aut</u> $\mathcal{O}$ is <u>true</u> auts of $\mathcal{O}$ - a group ind-scheme:
$(\underline{\mathrm{Aut}}\ \mathcal{O})(R) := \mathrm{Aut}^{topol}_R R[[t]]$ topological autos.
$t \mapsto a_0 + a_1 t + a_2 t^2 \ldots$, a_0 nilpotent & a_1 invertible.
- ind scheme since degree of nilpotence isn't fixed.
$\mathrm{Aut}^0 \mathcal{O} < \underline{\mathrm{Aut}}\ \mathcal{O}$ as set $a_0 = 0$...
can identify $\mathrm{Aut}^0\mathcal{O}$ with its <u>points</u>, autos of $\mathcal{O}$.
$\{t \mapsto t + a_0,\ a_0 \text{ nilpotent}\} \simeq \widehat{\mathbb{G}_a}$
<u>Aut</u> $\mathcal{O}$ as space is $\widehat{\mathbb{G}_a} \times \mathrm{Aut}\ \mathcal{O}$.

Lie $\underline{\mathrm{Aut}\,\mathcal{O}} = \mathrm{Der}\,\mathcal{O}$, Lie $\mathrm{Aut}^\circ \mathcal{O} = \mathrm{Der}^\circ \mathcal{O}$, vector fields vanishing at 0

For small g those properties characterize φ uniquely but not in general - $A(\mathcal{O})$ has automorphisms.

$x \in X \quad f_x : A(\mathcal{O}_x) = \mathfrak{Z}(\mathcal{O}_x) \to H^0(\mathrm{Bun}_G, D')$

$\downarrow$

$A = \mathrm{Fun}(\mathrm{Op}_{{}^L\mathfrak{g}}(X)) \qquad \mathrm{Op}_{{}^L\mathfrak{g}}(X) \hookrightarrow \mathrm{Op}_{{}^L\mathfrak{g}}(\mathcal{O}_x)$

Theorem $f_x : A(\mathcal{O}_x) \to H^0(\mathrm{Bun}_G, D')$ comes from $f : A \to H^0(\mathrm{Bun}_G, D')$, independent of x.

Plan of proof: 1. Construct an $\mathcal{O}_X$-algebra $\mathfrak{Z}_X$, (loc. free $\mathcal{O}_X$-module) with connection $\nabla : \mathfrak{Z}_X \to \mathfrak{Z}_X \otimes \omega_X$, such that $\mathfrak{Z}_x$ = fiber of $\mathfrak{Z}_X$ at x.

Write $\mathrm{Diff} = H^0(\mathrm{Bun}_G, D')$

2. $f_x : \mathfrak{Z}_x \to \mathrm{Diff}$ comes from $f : \mathfrak{Z}_X \to \mathcal{O}_X \otimes \mathrm{Diff}$ and f is horizontal

3. $\forall$ associative $\mathcal{C}$, $\mathrm{Hom}^\nabla(\mathfrak{Z}_X, \mathcal{O}_X \otimes \mathcal{C}) = \mathrm{Hom}(A, \mathcal{C})$ ($\mathrm{Spec}\,A = \mathrm{Op}_{{}^L\mathfrak{g}}(X)$].

1. Construct a tensor functor $\{\underline{\mathrm{Aut}\,\mathcal{O}}\text{-modules}\} \to \{\mathcal{O}_X \text{ mod w/connection}\}$ $W \mapsto W_X$. Take $W = \mathfrak{Z}(\mathcal{O})$ ...

To just get corresponding $\mathcal{O}_X$-module from $\mathrm{Aut}^\circ\mathcal{O}$-module take $\hat{X} = \{(x \in X, \mathcal{O}_x \xrightarrow{\sim} \mathcal{O})\}$ principal $\mathrm{Aut}^\circ\mathcal{O}$-bundle.

$\hat{X}$
$\downarrow$
X

As $\mathcal{O}_X$-module $W_X = \hat{X}$-twist of the $\mathrm{Aut}^\circ\mathcal{O}$-module W.

(integrable) Connection on bundle - identify infinitely close fibers: look at R-points of X.

$x_1, x_2 \in X(R)$ infinitely close if have equal images in $X(R_{red})$ $\quad R_{red} = R/\text{nilradical}$.

$\Leftrightarrow (x_1, x_2) : \mathrm{Spec}\,R \to X \times X$, $\searrow \Delta^{(n)} \nearrow$ image in n^{th} infi. neighborhood of $\Delta \hookrightarrow X \times X$ for some n.

__Lemma__ Infinitely close points $x_1, x_2 \in X(R)$ have equal formal neighborhoods (identify points with graph of maps $\operatorname{Spec} R \to X$, they'll have same formal completions.)

For $x_1, x_2 \in X(R)$ infinitely close define an isomorphism $(\mathcal{W}_X)_{x_1} \xrightarrow{\sim} (\mathcal{W}_X)_{x_2}$ of R-modules:

$\Rightarrow$ lift $\hat{x}_i \in \hat{X}(R)$, $\hat{x}_i \mapsto x_i$ (e.g. choose local coord near x_i). $\hat{x}_i$ defines isom $(\mathcal{W}_X)_{x_i} \xrightarrow{\sim} \mathcal{W} \otimes R$

so we need an automorphism of $\mathcal{W} \otimes R$ ...

$\hat{x}_i$ defines $\{\text{formal neigh. of } x_1\} \xrightarrow{\sim} \operatorname{Spec} R[[t]]$

$\{\text{formal neigh of } x_2\} \xrightarrow{\sim} \operatorname{Spec} R[[t]]$ $\circlearrowleft \gamma$

get $\gamma \in \operatorname{Aut} R[[t]] = (\underline{\operatorname{Aut}}\, \mathcal{O})(R)$, as required ...

$\underline{\operatorname{Aut}}\, \mathcal{O}$ acts on $\mathcal{W}$ $\Rightarrow$ γ defines the automorphism of $\mathcal{W} \otimes R$.

3 a). A ∇-invariant commutative subalgebra of $\mathcal{O}_X \otimes \mathcal{C}$ is $\mathcal{O}_X \otimes \mathcal{C}'$, $\mathcal{C}' \subset \mathcal{C}$ commutative subalgebra.

$\Rightarrow$ reduces 3 to case $\mathcal{C}$ commutative.

b.) Assume $\mathcal{C}$ commutative. $\mathcal{O}_X$-algebra $\leftrightarrow$ affine scheme $/X$. $\operatorname{Hom}_{\mathcal{O}_X\text{-alg}}(\mathfrak{Z}_X, \mathcal{O}_X \otimes \mathcal{C})$

$= \operatorname{Mor}_X(X \times \operatorname{Spec} \mathcal{C}, \operatorname{Spec} \mathfrak{Z}_X)$

$= \operatorname{Mor}_X(\operatorname{Spec} \mathcal{C}, \Gamma(\operatorname{Spec} \mathfrak{Z}_X))$

$\operatorname{Hom}^{\nabla}_{\text{alg}}(\mathfrak{Z}_X, \mathcal{O}_X \otimes \mathcal{C}) = \operatorname{Mor}(\operatorname{Spec} \mathcal{C}, \Gamma^{\nabla}(X, \operatorname{Spec} \mathfrak{Z}_X))$

$\parallel$ Now recall $\mathfrak{Z}(\mathcal{O}) = A(\mathcal{O}) \leadsto \mathfrak{Z}_X \xrightarrow{\sim} A_X$.

$\operatorname{Mor}_X(\operatorname{Spec} \mathcal{C}, \Gamma^{\nabla}(X, \operatorname{Spec} A_X))$

$(\operatorname{Spec} A_X)_x = \operatorname{Spec} A(\mathcal{O}_x) = \operatorname{Op}_{{}^LG}(\mathcal{O}_x)$

$\Rightarrow \operatorname{Spec} A_X = \{\text{jets of opers}\}$ w/ usual jet connection.

$\Rightarrow \Gamma^{\nabla}(X, \operatorname{Spec} A_X) = \operatorname{Op}_{{}^LG}(X)$ global opers.

2. Define $M \to X$, $M_x = \operatorname{Bun}_{G,\underline{x}}$.

Thus $M = \{x \in X, \mathcal{F}, \text{ triv. of } \mathcal{F} \text{ over } \operatorname{Spec} \mathcal{O}_x\}$

$J(G)$ = jets of functions $X \to G$, ~~a~~ scheme over X, fiber $G(\mathcal{O}_x)$.

$J^{mer}(G)$ · ind scheme w/ fiber ~~$G(\mathcal{K}_x)$~~. $G_{\mathcal{K}_x}$

Forget twisting of diffops for now... existence of connections $f: \mathfrak{Z}_X \to \mathcal{O}_X \otimes \mathcal{D}iff$ follows from above construction.

!!! M, $J(G)$, $J^{mer}(G)$ are X-schemes $^{-(ind)}$, with connections along X, and the action $J^{mer}(G) \times_X M \to M$ is horizontal, (as is $J^{mer}(G) \times_X J^{mer}(G) \to J^{mer}(G)$).

To see these, use language of infin. closed points...

Consider $M \to Bun_G \ni \mathcal{F}$, fiber $M_{\mathcal{F}}$ has connection

$M_{\mathcal{F}} = \{x \in X,$ triv of $\mathcal{F}$ over ~~Spec~~ $\hat{\mathcal{O}}_x \Leftrightarrow$ section of corresp principal bundle $\}$

$\Rightarrow M_{\mathcal{F}} =$ jets of sections of $\mathcal{F}$ (as principal bundle).