V. Drinfeld — DG-Categories — 10/24/02

Drawbacks with triangulated categories! as basic but not as good as Hilbert spaces...
DG & A_∞ category ... practically equivalent
richer structures, ~~also~~ simpler, come naturally in derived category context.

k commutative ring
Complex = complex of k-modules $\ldots \to C^{i-1} \to C^i \xrightarrow{d} C^{i+1} \to \ldots$
usual picture of complex... alternatively $C = \bigoplus_i C^i$,
$d : C \to C$ deg $C = 1$, $d^2 = 0$:
i.e. C is a DG k-module : better POV psychologically!
complexes form a tensor category (symmetric monoidal category)
$\otimes$: $\{k\text{-DG-mod}\} \times \{k\text{-DG-mod}\} \longrightarrow \{k\text{-DG-mod}\}$
simpler to write in DG language:
$C \otimes C'$ usual $\otimes$ on underlying k-modules with sum of gradings, differential using (graded) Leibniz formula
... think uniformly in language of $\otimes$ categories rather than in "complex" picture.

$(A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$, $A \otimes B \xrightarrow{\sim} B \otimes A$ functorial
isoms satisfying identities.
Associativity for complexes obvious, commutativity is super-commutativity
homog elts $a \otimes b \longmapsto (-1)^{pq}\, b \otimes a$ $p = \deg a$, $q = \deg b$
k = unit object.

Def DG-category = category enriched over k-DG-modules.
$\mathcal{A}$ ~~category~~ has objects $\mathrm{Ob}\,\mathcal{A}$,
& $\mathrm{Hom}(X, Y) \in$ k-DG-mod morphism
composition $\mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$ associative.
Identity : $X \in \mathrm{Ob}\,\mathcal{A} \Rightarrow$ morphism $k \longrightarrow \mathrm{Hom}(X,X)$

map ~~$k \to C$~~ $\Leftrightarrow$ element $z \in C$ with deg $z = 0$ & $dz = 0$
so $id_X \in \mathrm{Hom}(X,X)$ deg 0 closed.

Can consider ~~as~~ plain category: forget grading & differential
$\to$ k-linear category

Another categorical way to pass from enriched to usual category: can replace $\mathcal{C}$ by set $Hom(k, \mathcal{C})$ = cocycles of degree zero ...

Examples: k-DG-mod is a DG-category: $Hom(\mathcal{C}, \mathcal{C}')$ has natural DG-mod: grading comes from natural $\mathbb{G}_m$ action on Hom....
replace whole Hom by sum of homogeneous submodules (in graded end) $Hom(\mathcal{C}, \mathcal{C}')^n = \prod_k Hom(\mathcal{C}^k, \mathcal{C}'^{k+n})$

Morphism of complexes is a 0-cocycle in this dg hom $Hom(\mathcal{C}, \mathcal{C}')$ ie. $Hom_{complexes}(\mathcal{C}, \mathcal{C}') = Hom(k, \underline{Hom}(\mathcal{C}, \mathcal{C}'))$

So DG-categories are categories with dg k-mod structure on hom sets so that compositions are k-linear.
Where does commutativity of $\otimes$ come in? $\rightsquigarrow$
can speak of dual category of DG category
$\mathcal{A}^o$ = dual DG category

Example 1. DG category with one object $\longleftrightarrow$ DG-algebra (unital - implicit henceforth)
2. A usual k-linear category = DG category with Homs are deg 0, $d=0$.
3. Complexes of R-modules, R an algebra
3'. Complexes of $\mathcal{R}$-modules, $\mathcal{R}$ a k-linear category:
$\mathcal{R}$-module = k-linear functor $\mathcal{R} \longrightarrow k$-mod
- psychologically important · think of more concretely & contained...
e.g. left A-mod $\otimes$ right A-mod $\Rightarrow$ group / bimodule
~ same construction for $\mathcal{R}$-modules
e.g. $X_1, \dots X_n \in \mathcal{R}$ fin many objects s.t. each object of $\mathcal{R}$ is isomorphic to a direct sum (finite) of these
$\rightarrow$ $\mathcal{R}$-module is module over $R = End \oplus X_i$ formal sum.

Even if $\mathcal{R}$ arbitrary can form formal direct sum $\bigoplus_{X,Y} Hom(X,Y)$ - get algebra but without unit, only many idempotents

4. DG-modules over a DG-algebra $R = \oplus R_k$
CQ: special case of 3' for $\mathcal{R} = \quad \cdot \quad \cdot \xrightarrow{R_k} \cdot \quad \cdot$

Another POV: a quasifunctor is a <u>bimodule</u> over A_1, A_2 with certain properties..
Another POV: A_∞-functors between DG categories...
i.e. replace A by a certain (canonical) resolution
– at least over a field!

<u>Triangulated Categories</u>

Example $K(C)$ = homotopy category of complexes in a k-linear category

A triangulated category is a graded category with additional structure: A "candidate triangle" is
$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "g"] \\ & Z \arrow[ul, "h"] \end{tikzcd}$$
$\deg f = \deg g = 0$, $\deg h = 1$, $gf = 0 = hg = fh$

Additional structure: some candidate Δ's are "distinguished" + axioms..

Don't require <u>shift</u> as structure: ~~only~~ in this graded setting ask for representability of suspension:

<u>Axiom 0</u> $\forall X \in C$ $\forall n \in \mathbb{Z}$ $\exists Y \xrightarrow{f} X$ $\deg f = n$ s.t. f^{-1} exists.

$\Rightarrow$ defined up to unique <u>isomorphism</u>, call $Y = \Sigma^n X$

$\Rightarrow$ weak action of $\mathbb{Z}$ on C.

$K(C)$: distinguished Δ is isom to $X \xrightarrow{f} Y \to \mathrm{Cone}\, f \to X[1]$

$(\mathrm{Cone}\, f)^n = Y^n \oplus X^{n+1}$ $\quad d_{C(f)} = \begin{pmatrix} d_Y & f \\ 0 & -d_X \end{pmatrix}$

$d_{X[1]} = -d_X$

Another POV (Deligne):
$$\begin{tikzcd} \cdots \arrow[r] & Y^{n-1} \arrow[r] & Y^n \arrow[r] & Y^{n+1} \arrow[r] & \cdots & \deg 0 \\ & X^{n-1} \arrow[u] \arrow[r] & Y^n \arrow[u, "f"] \arrow[r] & X^{n+1} \arrow[u] \arrow[r] & X^{n+2} & \deg 1 \end{tikzcd}$$

– forms a double complex (almost), & cone is total complex ... really <u>naive</u> double complex!

$df = fd$ ~~no signs~~ no signs so don't get $(d' + d'')^2 = 0$, so must introduce signs

<u>Examples</u> a. $K(C)$ eg $K(R\text{-mod})$ R k-algebra

b. $Ho^{\bullet}(R\text{-DG mod})$, R a DG algebra or

c. $Ho^{\bullet}(\mathcal{R}\text{-DG mod})$

$\mathcal{T}$ a triangulated category $\supset \mathcal{T}'$ a full subcategory.
a candidate triangle in $\mathcal{T}'$ is distinguished if it is so in $\mathcal{T}$
... ask these to give Δ structure on $\mathcal{T}'$ $\Rightarrow$ triangulated subcategory
$\Leftrightarrow$ $\mathcal{T}'$ closed wrt suspensions, desuspensions & cones
in weak sense

Advantage of DG categories: Yoneda $\mathcal{A} \hookrightarrow \mathcal{A}^{\circ}$-DG-mod
Triangulated case have $\mathcal{T} \hookrightarrow$ ContraFunctors $\mathcal{T} \to$ graded modules

and in fact $\mathcal{T} \hookrightarrow$ cohomological functors, i.e. graded functors
$\mathcal{T}^{\circ} \longrightarrow$ Graded k-modules. However cohomological functors
don't have natural triangulated structure...

DG categories II

$\mathcal{A}$ DG-category $\Rightarrow$ homotopy category $Ho^{\bullet}(\mathcal{A})$
$\mathcal{A}_1 \longrightarrow \mathcal{A}_2$ DG functor is a quasi-equivalence if induces
equivalence $Ho^{\bullet}(\mathcal{A}_1) \longrightarrow Ho^{\bullet}(\mathcal{A}_2)$.
Would like to localize "world" of DG categories wrt quasi-equivalences..

Triangulated category: $\begin{matrix}\text{graded category +}\\ \text{additional structure}\end{matrix}$, class of distinguished triangles.

Yoneda : $\mathcal{A} \underset{full}{\hookrightarrow} \mathcal{A}^{\circ}-DG\text{-mod}$, $\quad Ho^{\bullet}(\mathcal{A}) \hookrightarrow Ho^{\bullet}(\mathcal{A}^{\circ}\text{-DG-mod})$
has triangulated structure

Def A candidate triangle in $Ho^{\bullet}(\mathcal{A})$ is said to be distinguished if it is distinguished in $Ho^{\bullet}(\mathcal{A}^{\circ}\text{-DG-mod})$
$\mathcal{A}$ is said to be pre-triangulated if $Ho^{\bullet}(\mathcal{A})$ is triangulated when equipped with this structure, i.e if $Ho^{\bullet}(\mathcal{A})$ is a triangulated subcategory of $Ho^{\bullet}(\mathcal{A}^{\circ}\text{-DG-mod})$, — i.e closed under cones & desuspensions.

Problem: this def is not self-dual: in $Ho^{\bullet}\mathcal{A}^{\circ}$ get two notions of distinguished triangle, opposite to each other. not clear if they agree. Do DG functors preserve these distinguished triangles?
$\rightsquigarrow$ more convenient to reformulate concretely etc

4' DG-modules over a DG-category $\mathcal{R}$ $\quad$ $\mathcal{R}$-DG-mod
(DG functor $\mathcal{R} \longrightarrow k$-DG-mod)

For any DG-category $\mathcal{R}$ have Yoneda embedding $\mathcal{R} \hookrightarrow \mathcal{R}^o$-DG-mod
(right $\mathcal{R}$-modules) $\quad X \longmapsto h_X$, $\quad$ contravariant

$h_X(Y) = Hom(Y,X)$

"embedding": fully faithful ($Hom(X,Y) \longrightarrow Hom(F(X),F(Y))$ isomorphism)

$\mathcal{A}$ = DG category, $Ho(\mathcal{A})$ = <u>homotopy category</u>:
Obs $Ho(\mathcal{A})$ = Obs $\mathcal{A}$
$X, Y \in$ Obs $\mathcal{A}$ $\quad$ $Ho(\mathcal{A})$-morphisms $X \to Y$ are $H^0 Hom(X,Y)$.
Other notation $Ho(\mathcal{A}) = H^0(\mathcal{A})$.

$Ho^\cdot(\mathcal{A})$ = graded category, the graded homotopy category, with morphisms elements of $\bigoplus_n H^n Hom(X,Y) =: Ext^n(X,Y)$
... can have <u>property</u> (not structure) of being triangulated ... if not can add some objects to make it triangulated.

<u>Def</u> A DG-functor $F: \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is a quasi equivalence if $Ho^\cdot(F): Ho^\cdot(\mathcal{A}_1) \longrightarrow Ho^\cdot(\mathcal{A}_2)$ is an equivalence, i.e.
1. $Hom(X,Y) \longrightarrow Hom(F(X),F(Y))$ is a quism
2. essential surjectivity of $Ho(F): Ho(\mathcal{A}_1) \longrightarrow Ho(\mathcal{A}_2)$ (nongraded version): $\forall Z \in \mathcal{A}_2$ $\exists X \in \mathcal{A}_1$, $f: F(X) \longrightarrow Z$ $\deg f = 0$, $df = 0$, f has a homotopy inverse.

$\mathcal{A}_1$, $\mathcal{A}_2$ DG-categories are quasiequivalent if
$\exists$ diagram of quasiequivalences

$$\mathcal{A}_1 \xleftarrow{\sim} \tilde{\mathcal{A}} \xrightarrow{\sim} \mathcal{A}_2$$

- Not enough to define correct structure on "world" of dg categories ... eg equivalence of quasiequivalences,
.. what do dg functors form?

"Fuzzy notion" of quasi functor $\mathcal{A}_1 \longrightarrow \mathcal{A}_2$:

$$\begin{tikzcd} \tilde{\mathcal{A}}_1 \arrow[d, "\text{quasi-equivalence}\ \wr"'] \arrow[dr, "\text{DG-functor}"] & \\ \mathcal{A} & \mathcal{A}_2 \end{tikzcd}$$

These form (following Kontsevich) a DG-category in A_∞ setting

Need for $\mathcal{A}^\circ$-DG-mod: carries cones of morphisms from $\mathcal{A}$.. $\mathrm{Cone}(f : X \to Y) = \mathrm{Cone}(h_X \xrightarrow{f} h_Y)$ on representable functors

— Don't need full $\mathcal{A}^\circ$-DG-mod — just cones (& iterated cones) & desuspensions, so don't need big $\mathcal{A}^\circ$-DG-mod.

$\mathcal{C}$ pre-additive category (have sums of morphisms but not nec direct sums) — to formally add direct sums can go several routes: $\mathcal{C} \hookrightarrow \mathcal{C}^\circ\text{-mod}$
$\hookrightarrow (\mathcal{C}\text{-mod})^\circ$ ∴ both have finite $\oplus$ can add them

Or concretely just look at category with objects $\bigoplus C_i$ $C_i \in \mathcal{C}$ formal direct sums, morphisms just matrices (rather than "linear transformations" in other defs).

"Def of "the pretriangulated hull"

Step 1 Replace $\mathcal{A}^\circ$-DG-mod by $\tilde{\mathcal{A}} \subset \mathcal{A}^\circ$-DG-mod full subcategory: smallest full DG-subcategory s.t. $\tilde{\mathcal{A}} \supset \mathcal{A}$ and
- $M \in \tilde{\mathcal{A}}$, $M' \simeq M$ isomorphism $\Rightarrow M' \in \tilde{\mathcal{A}}$
- $M \in \tilde{\mathcal{A}}$, $n \in \mathbb{Z} \Rightarrow M[n] \in \tilde{\mathcal{A}}$.
- M is stable wrt semisplit extensions: $0 \to M_1 \to M_2 \to M_3 \to 0$ is semisplit extension (of DG-modules) if split in sense of graded modules ... eg cone $0 \to N \to \mathrm{Cone}(M \to N) \to M[1] \to 0$ graded splitting
  - f determines failure of splitting to be closed — its differential is f.

More concretely $M \in \tilde{\mathcal{A}} \iff \exists\ 0 \subset M_1 \subset \dots \subset M_n = M$ finite filtration st $M_i / M_{i-1} \simeq h_{a_i}[r_i]$ $a_i \in \mathcal{A}$, $r_i \in \mathbb{Z}$
- representable up to shift. ($h_a \in \mathcal{A}^\circ$-DG-mod $\leftrightarrow a \in \mathcal{A}$)

... the extensions $0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$ are automatically semi split because representable functors are projective objects $h_a \in \mathcal{A}^\circ$-graded modules.

eg suppose $\mathcal{A} = A$ is an algebra (ie only one object)
— $a \in \mathcal{A}$! object $\Rightarrow$ module $h_a = A$ as A-module, which is free $\Rightarrow$ projective.

Step 2 Fix the splittings $0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$
So $M = \bigoplus h_{a_i}[r_i]$ as graded modules.

As a DG-module $M = (\bigoplus_i h_{a_i}[r_i] \quad \nabla_\omega = d + \omega)$

d = standard differential on direct sum of dg modules

$\omega = (\omega_{ij}) \qquad \omega_{ij} \in Hom(a_j, a_i)[r_i - r_j]$

of degree 1 (wrt grading)

1. ω is strictly upper triangular $\omega_{ij} = 0$ if $i \geq j$
2. $\nabla_\omega^2 = 0$ ie $\boxed{d\omega + \omega^2 = 0}$ flat connection. (Maurer-Cartan eqn).

Def $\mathcal{A}^{pretr}$ the pretriangulated hull of $\mathcal{A}$: "DG category of twisted complexes"

Ob $\mathcal{A}^{pretr}$ are formal expressions $(\hat{\bigoplus}_{i=1} a_i[r_i], \omega)$

$\omega = (\omega_{ij})$ as above.

$Hom(\oplus a_j[r_j], \oplus a_i'[r_i'])$ as graded module

(ie ignore d, ignore ω) is just space of matrices

$f = (f_{ij}) \qquad f_{ij} \in Hom(a_j, a_i')[r_i' - r_j]$.

composition automatic

$\tilde{d} f = \nabla_{\omega'} f - f \nabla_\omega (-1)^{\deg f}$ super commutator:

symbol ∇ has degree 1 $\qquad = d_{naive} + \omega' f - (-1)^{\deg f} f \omega$.

'twisted complexes' --- Toledo-Tong.

This definition is clearly self-dual, & clear that DG functors extend automatically to pretriangulated hull.

$f : X \to Y$ in $\mathcal{A}$ $\Rightarrow$ $Cone(f) \in \mathcal{A}^{pretr}$ canonical object

$Cone(f) = (Y \oplus X[1], \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix})$.

$X \in \mathcal{A}$, $X[n] \in \mathcal{A}^{pretr}$

Def $\begin{tikzcd} X \arrow[r] & Y \arrow[d] \\ & Z \arrow[ul] \end{tikzcd}$ in $Ho^\cdot(\mathcal{A})$ is distinguished if it is isomorphic to $\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d] \\ & Cone(f) \arrow[ul] \end{tikzcd}$

$\mathcal{A}$ is pretriangulated if $\forall f : X \to Y$ in $\mathcal{A}$, $Cone(f)$ is homotopy equivalent to an object of $\mathcal{A}$, & same for $X[n]$ $n \in \mathbb{Z}$.

Exercise 1. Notion of distinguished Δ is self dual
2. $F : \mathcal{A} \to \mathcal{B}$ DG-functor $\Rightarrow$ $Ho^\cdot F : Ho^\cdot \mathcal{A} \to Ho^\cdot \mathcal{B}$

sends distinguished triangles to distinguished triangles. If H^0F fully faithful $\Rightarrow$ converse also holds.

3. $\mathcal{A}^{pretr}$ is a pretriangulated DG-category
4. $H^0(\mathcal{A}^{pretr})$ is generated as Δ-category by $H^0(\mathcal{A})$
5. $\mathcal{A}$ is pretriangulated iff $\mathcal{A} \to \mathcal{A}^{pretr}$ is a quasiequivalence.
6. $\mathcal{A}^{pretr} \to (\mathcal{A}^{pretr})^{pretr}$ is an equivalence of DG-categories.
7. If $F: \mathcal{A} \to \mathcal{B}$ is a quasiequivalence then it induces a quasiequivalence $F^{pretr}: \mathcal{A}^{pretr} \to \mathcal{B}^{pretr}$

Bondal-Kapranov notation $\mathcal{A}^{tr} := H^0(\mathcal{A}^{pretr})$.

$\mathcal{A}$ DG-category $\rightarrow$ triangulated category.
Do derived categories, e.g. $\mathcal{D}(A\text{-mod})$ or $\mathcal{D}(\text{abelian category})$ come from a DG-category? abelian categories are complicated objects .... take another for

$\mathcal{A}$ a DG category $\Rightarrow \mathcal{D}(\mathcal{A}) :=$ derived category of $\mathcal{A}^0$-modules $:= H^0(\mathcal{A}^0\text{-DG-mod}) / (\text{acyclic DG } \mathcal{A}^0\text{-mod})$
- quotient of triangulated category by subcategory.

Importance of this example (B. Keller): have Yoneda $\mathcal{A} \hookrightarrow \mathcal{A}^0$ DG-mod extends $\Rightarrow$ fully faithful

$\mathcal{A} \hookrightarrow \mathcal{A}^{pretr} \to$, $\mathcal{A}^{tr} := H^0(\mathcal{A}^{pretr}) \hookrightarrow \mathcal{D}(\mathcal{A})$ fully faithful ... so any of the $\mathcal{A}^{tr}$'s embed into such a derived category of modules.

Exercise*: A quasiequivalence $\mathcal{A} \to \mathcal{B}$ induces an equivalence $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$: have restriction functor in one direction ~~$\mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$~~ & derived functor of induction

$\mathcal{D}(\mathcal{A}) \underset{Res}{\overset{L(Ind)}{\rightleftarrows}} \mathcal{D}(\mathcal{B})$ always adjoint, & in this case adj. morphisms are isomorphisms.

Q: $\mathcal{D}(\mathcal{A}) = H^0(?)$? DG-category containing $\mathcal{A}$.
Answer (known to topologists though question unexplained?) - Spaltenstein, Avramov-Halperin, Hinich

Will define $\underrightarrow{\mathcal{A}} \longrightarrow \mathcal{A}^{\circ}$-DG-mod : category of
<u>semifree</u> DG-modules (~ cell complexes ...)
so that $Ho^{\circ}(\underrightarrow{\mathcal{A}}) \longrightarrow \mathcal{D}(\mathcal{A})$ is an equivalence.

Well known (in <u>bounded</u> setting) : R a usual ring,
$\left\{ \begin{array}{l} \text{Derived category of bounded above} \\ \text{complexes of } R\text{-modules} \end{array} \right\} = \left\{ \begin{array}{l} \text{homotopy category of bounded above} \\ \text{complexes of projective modules} \end{array} \right\}$

try to eliminate boundedness :
<u>example</u> $R = \mathbb{Z}/4\mathbb{Z}$ (essential that R has ∞ homological dimension!)
$\cdots \to R \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} R \to \cdots$
acyclic complex but <u>not</u> homotopic to zero :
apply $\otimes_R \mathbb{Z}/2\mathbb{Z}$, preserves null-homotopy, but get
$\cdots \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to \cdots$
with nonzero cohomology!

So to eliminate boundedness need projectivity assumption on <u>whole</u> complex not its terms..

<u>Exercise</u> If A has finite homological dimension ($\exists$ finite proj resolutions)
then can remove boundedness in above ~~statem~~ equivalence.
(in particular acyclic complexes of projectives will be null homotopic)

<u>Def</u> A $\mathcal{DG}$ - $\mathcal{A}^{\circ}$-module P is <u>free</u> if it is $\mathcal{DG}$-isomorphic
to $\bigoplus h_{a_i}[r_i]$ (possibly infinite sum)
$\Longrightarrow$ projective
(in case of just a DG-algebra: module generated by e_i's of degree $-r_i$ with $de_i = 0$).

P is <u>semifree</u> if $\exists$ $0 \subset P_1 \subset P_2 \subset \cdots$ exhaustive filtration by DG-modules such that each quotient is free.
free $\Rightarrow$ projective so extensions are <u>semi</u>split. (not split for differential!)

$\tilde{\mathcal{A}} \subset \mathcal{A}^{\circ}$-DG-mod introduced before is just the DG-category
April 5/ of finitely generated semifree DG-modules!
So $\{$semifree modules$\} = \underrightarrow{\mathcal{A}}$ is infinite version of $\tilde{\mathcal{A}}$.

$\underset{\to}{A}$: $\overset{DG}{\text{analog of}}$ category of ind-objects (systems $\varinjlim X_i$ or certain A^0-modules.. representable functors)

I abstract category of indices, functor $I \to A$
$\Rightarrow$ consider homotopy colimit of this functor DG object associated to some simplicial object.. colimits of such & of weak (A_∞-) functors like in $\underset{\to}{A}$

– inspired by (generalized) cell complexes .. ie don't necessarily attach in correct order.

A DG-algebra. A DG-module P over A^0 is semifree if $\exists$ $0 \subset P_1 \subset P_2 \subset \ldots$ $\bigcup P_i = P$, P_i freely generated by P_{i-1} & homogeneous generators e_j, $j \in J_i$ so that $de_j \in P_{i-1}$.
(& $d_{P_i}|_{P_{i-1}} = d_{P_{i-1}}$)

of course $\Longleftrightarrow$ P_i/P_{i-1} is free as dg module.
Advantage of above formulation: semifreeness makes sense in nonlinear situation: eg for DG algebras
– same def: exists filtration, of course quotients won't be algebras .. $P_i = P_{i-1}\langle X_j \rangle$ freely generated by P_{i-1} & generators X_j, differentials are contained in subalgebra of old generators.
Same for algebras over any operad..

eg topological semigroups: attaching cell means add cell in topological sense (map from sphere), generate semigroup by old semigroup & this cell with attaching map. --- combination of algebraic free generation & topological adding cells

eg Stasheff operad: to what is it an answer? construct semifree resolution of the associative operad (in topological operads)

Theorem: $\exists$!) model structure on A^0 DG-mod (or on DG-alg or semifree operads or..) with W: ~~homotopy equivalences~~ quasi isomorphisms, F = surjections

The resulting cofibrant objects (ie morphisms $0 \to P$ are cofibrations) are retracts of semifree objects.
(linear case: retract ⇝ direct summand)

initial object

<u>Example 3</u> A bounded above complex of free R-modules is semifree
$. \to P^{n-2} \to P^{n-1} \to P^n \to 0$; $P = \oplus P^i$, with stupid filtration
--- so theorem about bounded above complexes is special case.

Example 4 $\to \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \ldots$
is not semifree (can't find basis vector annihilated by the differential!)

<u>Theorem</u> $Ho^\bullet(\underrightarrow{A}) \longrightarrow \mathcal{D}(A)$ is an equivalence.

<u>Proof</u> In bounded above setting: use Lemma 1: everything has bounded above projective resolution. Lemma 2: morphism from bounded above projective to acyclic is null homotopic ...

<u>Lemma 1</u> $\forall M \in A^\circ\text{-DG-mod}$ $\exists$ quism $P \to M$, $P \in \underrightarrow{A}$
(in fact can choose <u>surjective</u> - though not necessary).
--- a model category axiom!
Lemma 2 $P \in \underrightarrow{A} \Rightarrow \forall$ acyclic $M \in A^\circ\text{-DG-mod}$, every morphism $P \to M$ is homotopic to 0
-- ie P is "<u>K-projective</u>" or <u>homotopically projective</u>.

<u>Exercise</u> M is homotopically projective iff it is homotopy equivalent to a semifree module.

Pf of <u>Lemma 2</u> : $P = \bigcup_{i=1}^{\infty} P_i$ P_i/P_{i-1} free. $f: M \to P$
i.e $f \in Hom(P, M)$ $\deg f = 0$, $df = 0$

Construct homotopy: i.e $f = dH$, $H \in Hom(P, M)$ $\deg H = -1$
-- build inductively, $H_i \in Hom(P_i, M)$ $\deg -1$,
$dH_i = f_i = f|P_i$.

H_i given H_{i-1}! choose s.t. $\tilde{H}_i|_{P_{i-1}} = H_{i-1}$, $\tilde{H}_i$ deg -1
correct $H_i = \tilde{H}_i + \bar{H}_i$, $\bar{H}_i \in Hom(P_i/P_{i-1}, M)$ deg -1

s.t. $d\tilde{H}_i = f_i - d\tilde{\tilde{H}}_i \in \mathrm{Hom}(P_i/P_{i-1}, M)$, $d(f_i - d\tilde{\tilde{H}}_i) = 0$

But this DG-module of Hom's is acyclic:
A/P_{i-1} is a direct sum of free (shifted) modules, so Hom is a product of shifted copies of M $\Rightarrow$ acyclic ∎

Pf of Lemma: Sublemma $N \to M$ morphism of DG-modules
$\Rightarrow$ $\exists$ factorization $N \hookrightarrow N' \to M$
s.t. 1. ~~N' is acyclic~~ $H^\cdot(N') \to H^\cdot(M)$ bijective
2. N'/N semifree

Sublemma $\Rightarrow$ lemma: construct surjective $f: P_1 \to M$
P_1 semifree & $H^\cdot P_1 \twoheadrightarrow H^\cdot M$
- then apply sublemma for $N = P_1$ $\Rightarrow$ $P \to M$
surjective & isom on cohomology:
$m \in M$ a generator, make module with two generators
e, e' of correct degree, $de = e'$, $de' = 0$.
$\to$ surjectivity on module. similarly for cohomology
- lift to cocycles.

Proof of sublemma: enough to construct $N \hookrightarrow N_1 \to M$, N_1/N semifree
with $\mathrm{Ker}(H^\cdot N \to H^\cdot M) = \mathrm{Ker}(H^\cdot N \to H^\cdot N_1)$ ∎

--- then build $N' = \bigcup N_i$ where $N \hookrightarrow N_1 \hookrightarrow N_2 \hookrightarrow \; \to M$

$K := \mathrm{Ker}(H^\cdot N \to H^\cdot M)$. $h_i \in K$ homog generators ($H^\cdot A$ module)
... add cells to kill cohomology

V. Drinfeld – DG Categories III 11/11/02

- 2-category of DG-categories … partial answer to "What do DG-categories form?" … really they form some kind of ∞-category.

Today: All DG categories are pretriangulated unless noted otherwise

Typical example of 2-category: Cat – have objects: categories, morphisms: functors, 2-morphisms: natural transformations

We'll define 2-category DG-cat. Objects: (small) DG-categories
$\mathcal{A}, \mathcal{B} \in$ DG-cat $\Rightarrow$ $T(\mathcal{A}, \mathcal{B})$ category of quasifunctors. Ob $T(\mathcal{A}, \mathcal{B})$ are 1-morphisms of our 2-category, morphisms in $T(\mathcal{A}, \mathcal{B})$ are 2-morphisms of DG-cat.

Composition is functor $T(\mathcal{A}, \mathcal{B}) \times T(\mathcal{B}, \mathcal{C}) \longrightarrow T(\mathcal{A}, \mathcal{C})$
Associativity here is structure not property:
weak associativity $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C} \xrightarrow{\phi} \mathcal{D}$
is an isomorphism $((\phi G) F) \xrightarrow{\sim} (\phi (GF))$ functorial in all three – associativity constraint
+ Pentagon axiom for composable 1-morphisms

$$((F_1 F_2) F_3) F_4 \qquad (F_1 (F_2 F_3)) F_4$$
$$(F_1 F_2)(F_3 F_4) \qquad F_1((F_2 F_3) F_4)$$
$$F_1(F_2(F_3 F_4))$$

$\forall \mathcal{A} \in$ DG-Cat $\exists\, id_{\mathcal{A}} \in T(\mathcal{A}, \mathcal{A})$ s.t.
$\exists$ isomorphism $id_{\mathcal{A}} \circ F \xrightarrow{\sim} F$ functorial wrt F
& similar for $F \circ id_{\mathcal{A}} \to F$

Here $T(\mathcal{A}, \mathcal{B})$ will in fact be triangulated
– Problem: should really replace $T(\mathcal{A}, \mathcal{B})$ by DG-category!
Possible over a field, unclear over ring.

$\mathcal{A} \in$ DG Cat $\longmapsto Ho(\mathcal{A})$ triangulated category
will give a 2-functor Ho : DG-Cat $\longrightarrow$ Tri Cat

Stupid 2-category : $\underline{DG\text{-}Cat}^{\text{naive}}$: Objects are DGcat.
1-morphisms are DG-functors, 2-morphisms are closed morphisms of DG-functors : $F, G : \mathcal{A} \to \mathcal{B}$
a closed morphism is $\eta : F \to G$ s.t. $X \in \mathcal{A}$ $\eta_X : F(X) \to G(X)$ is closed.

Problems: want to invert quasiequivalence... also this notion of closed morphism not invariant wrt quasiequivalence.

Ex. $\mathcal{A}$ arbitrary DG-category, k ground field considered as DG-algebra or DG category with one object
$T(k, \mathcal{A}) \; (= T(k^{\text{pretr}}, \mathcal{A})) = ?$

Answer $= Ho(\mathcal{A})$!

(k^{pretr} : add direct sums and add cones for morphisms ... ie $k^{\text{pretr}} = \{$finite complexes of f.gen free k-modules$\}$)

Ex. $T(\mathcal{A}, \underrightarrow{k})$ $\quad \underrightarrow{k}$ = ind version of k^{pretr}

$\underrightarrow{k}$ = semifree complexes

Answer: $T(\mathcal{A}, \underrightarrow{k})$ = derived category of $\mathcal{A}$-modules

(recall $\mathcal{A}$-module is just DG-functor $\mathcal{A} \to$ complexes).
Here get derived not homotopy category ...
... doesn't change under quasiequivalence unlike homotopy category

$= Ho(\underrightarrow{\mathcal{A}^\circ})$ $\quad$ left semifree $\mathcal{A}$-modules
$=: \mathcal{D}(\mathcal{A})$

Ex. $T(\mathcal{A}, k^{\text{pretr}}) \subset \mathcal{D}(\mathcal{A})$ full subcategory
$= \{F : \mathcal{A} \to$ Complexes s.t. $\forall a \in \mathcal{A}$, $F(a)$ is quasiisomorphic to a finite complex of free k-modules of finite rank $\in k^{\text{pretr}}\}$

<u>Keller's def</u> $\quad k$ = field
$\underrightarrow{T}(\mathcal{A}, \mathcal{B}) := \mathcal{D}(\mathcal{A} \otimes \mathcal{B}^\circ)$ derived category of bimodules
..."ind quasifunctors"

$M \in \underrightarrow{T}(\mathcal{A}, \mathcal{B})$ gives functor $F_M : Ho(\mathcal{A}) \to \mathcal{D}(\mathcal{B}^\circ)$
$= Ho(\mathcal{B})$

$T(\mathcal{A},\mathcal{B}) = \{ M \in \underrightarrow{I}(\mathcal{A},\mathcal{B}) : \forall a \in Ob(\mathcal{A})\ \Phi_M(a)$ is isomorphic to an object of $Ho(\mathcal{B}) \}$

.. can define $\mathcal{D}G\,cat \subset \underrightarrow{\mathcal{D}G\,cat}$ where 1-morphisms are ind quasifunctors in $\underrightarrow{I}(\mathcal{A},\mathcal{B})$ ....

Composition : $M \in \underrightarrow{I}(\mathcal{A},\mathcal{B}) = \mathcal{D}(\mathcal{A} \otimes \mathcal{B}^\circ)$, $N \in \mathcal{D}(\mathcal{B} \otimes \mathcal{C}^\circ)$
$\Rightarrow M \overset{L}{\otimes}_{\mathcal{B}} N \in \mathcal{D}(\mathcal{A} \otimes \mathcal{C}^\circ)$

Note : over a field can lift everything from $\mathcal{D}G$ to triangulated .... can choose all associators trivial over a field ! M, N semifree then don't need to derive $M \overset{L}{\otimes}_{\mathcal{B}} N = M \otimes_{\mathcal{B}} N$ & is semifree.

Not over field : only $\overset{L}{\otimes}$ will be well defined under quasiequivalence (without some flatness assumptions...)

Any k : everything works under additional assumption of 'flatness':

Def $\mathcal{A}$ is homotopically flat / k if all Hom complexes are homotopically flat.

Def A complex $C^\cdot$ is said to be homotopically flat if $C^\cdot \otimes$ (acyclic) is acyclic.

Homotopical Flatness & Projectivity Recall M is projective if
Hom $(M, -)$ is exact,
complex C is homotopically projective if (dg) functor Hom $(P, -)$ preserves acyclicity $\Leftrightarrow$ preserves quasi isomorphisms

- A semifree complex is homotopically flat ($\varinjlim$ of f.g. semifree modules, $\otimes$ commutes with filtering direct limits)
- Show homotopical projectivity $\Rightarrow$ homotopical flatness
- suppose C is homotopically flat & acyclic $\Longrightarrow$ $C \otimes$ (any complex) is acyclic ...

sometimes better to have all hom modules semifree —
even better behaved than homotopically flat ..

Any DG-category has a semifree resolution, hence
in particular a homotopically flat resolution

$\mathcal{A}$ DG-category $\Rightarrow$ class of homotopically flat
resolutions $\tilde{\mathcal{A}}_i \longrightarrow \mathcal{A}$, $i \in I$
- need to compare categories $T(\tilde{\mathcal{A}}_i, \mathcal{B})$
(don't need to resolve both $\mathcal{A}, \mathcal{B}$) ...

$$\mathrm{Ob}\, T(\mathcal{A}, \mathcal{B}) = \coprod_{\tilde{\mathcal{A}}_i \to \mathcal{A}} T(\tilde{\mathcal{A}}_i, \mathcal{B})$$

- don't have to be stingy in defining objects in category
since only care about category up to equivalence

or even $\coprod_{\substack{\tilde{\mathcal{A}}_i \to \mathcal{A} \\ \tilde{\mathcal{B}}_j \to \mathcal{B}}} T(\tilde{\mathcal{A}}_i, \tilde{\mathcal{B}}_j)$.

$\forall i,j \in I$ want canonical object $F_{ij} \in T(\tilde{\mathcal{A}}_i, \tilde{\mathcal{A}}_j)$
— "identity on $\mathcal{A}$", with compatibility
$i\ j\ k \in I$ want $F_{ij} F_{jk} \xrightarrow{\sim} F_{ik}$ & inverses.

$$T(\tilde{\mathcal{A}}_i, \tilde{\mathcal{A}}_j) \subset \mathcal{I}(\tilde{\mathcal{A}}_i, \tilde{\mathcal{A}}_j) = \mathcal{D}(\tilde{\mathcal{A}}_i \otimes \mathcal{A}_j^{\circ})$$

But $\tilde{\mathcal{A}}_i \xrightarrow{\pi_i} \mathcal{A} \xleftarrow{\pi_j} \tilde{\mathcal{A}}_j$ $\Rightarrow$ can send $\tilde{\mathcal{A}}_i \times \tilde{\mathcal{A}}_j^{\circ} \longrightarrow$ complexes
$(a \in \tilde{\mathcal{A}}_i, a' \in \tilde{\mathcal{A}}_j) \longmapsto \mathrm{Hom}(\pi_j(a'), \pi_i(a))$

$\Rightarrow$ canonical F_{ij}.
So this allows us to define DG-cat without
imposing flatness, which is often unavailable
- eg dg(complexes of abelian groups) $/\mathbb{Z}$.

Problem doing all this on DG level:
get not concrete complexes but complexes up to
canonical homotopy equivalence $\dashrightarrow$ should loosen
DG category, to 2 objects assign many models
of hom complexes, & contractible family of homotopy

equivalences --- even weaker than A_∞ category ... some kind of ∞-category ... so for now $T(\mathcal{A}, B)$ is only triangulated not DG.

Note: category $\mathcal{A}$ is ~~qu~~ equivalent to category of pairs $X, Y \in \mathcal{A}$ and map $X \to Y$, which is quasi ~~equi~~ isomorphism — i.e. homotopy diagonal of $\mathcal{A}$.
Each of the two projections is a homotopy equivalence with $\mathcal{A}$.

Kontsevich model

There is a canonical 2-functor $\mathcal{DG}Cat^{nait} \longrightarrow \mathcal{DG}Cat$ (assume only flat versions for simplicity).
$F : \mathcal{A} \to \mathcal{B}$ DG-functor $\Rightarrow$ bimodule $M_F \in \mathcal{D}(\mathcal{A} \otimes \mathcal{B}^o)$;

analog of bimodule assoc to algebra hom $A \xrightarrow{f} B$
-- namely B is a $(A \otimes B^o)$-module.
-- in categorical version, $M_F : \mathcal{A} \times \mathcal{B}^o \to$ Complexes
is $M_F(a, b) = \mathrm{Hom}(b, F(a))$.
which actually lies in $T(\mathcal{A}, \mathcal{B}) \subset \underrightarrow{T}(\mathcal{A}, \mathcal{B}) = \mathcal{D}(\mathcal{A} \otimes \mathcal{B}^o)$

and the functor $H_0(\mathcal{A}) \longrightarrow H_0(\mathcal{B})$ coming from M_F is just $H_0(F)$
.SO we have the 2-functor on morphisms

Sasha: any quasifunctor can be viewed as functor
$\mathcal{A} \longrightarrow$ {rt $\mathcal{B}$-modules which are quasi-representable: quism to representable one}
$\mathcal{B}$

-ie localization of functors $\mathcal{A} \to \mathcal{B}$ ...

$\mathrm{Ext}^i_{\mathcal{A} \otimes \mathcal{B}^o}(M_F, M_G) = \mathrm{Ext}^i_{\mathcal{A} \otimes \mathcal{A}^o}(\mathcal{H}om_{\mathcal{A}}, \mathcal{H}om(F, G))$

Here $\mathcal{H}om(F, G)$ is the DG $\mathcal{A} \otimes \mathcal{A}^o$-module
$(a, a') \mapsto \mathrm{Hom}(F(a'), G(a))$

What is $\mathcal{H}om_{\mathcal{A}}$? for algebra A can think of A as an A-bimodule ...
$\mathcal{H}om_{\mathcal{A}}$ is the categorical analog: DG-$\mathcal{A}\otimes\mathcal{A}^{op}$-module
$(a, a') \mapsto \mathrm{Hom}(a', a)$ (i.e. $\mathcal{H}om(Id, Id)$)

$\rightsquigarrow$ gives 2-morphisms for our $DGCat^{naive} \to DGCat$
Write down composits.

[This gives the full subcategory of quasifunctors given by honest functors !
via $F \longmapsto M_F$ bimodules..
This subcategory is manifestly self-dual, unlike Keller equivalence definition.

Ex If $\mathcal{A}$ is semifree then $\forall$ objects of $T(\mathcal{A}, \mathcal{B})$ is isomorphic to some M_F.

Note $\mathcal{H}om_{\mathcal{A}}$ = diagonal bimodule

$Ext^i(\mathcal{H}om_{\mathcal{A}}, -)$ = Hochschild cohomology !
- gives full description of DG Cat for $\mathcal{A}$ semifree by above exercise... in general must use semifree resolutions.

If $\mathcal{A}$ is semifree module, i.e., Hom complexes of $\mathcal{A}$ are semifree $\Rightarrow$ $\mathcal{A}$ has a semifree resolution

eg k=field: Standard($\mathcal{A}$) bar (cobar) - most economic general resolution.

Can compute $T(\mathcal{A}, \mathcal{B})$ in terms of this resolution Stand($\mathcal{A}$):
Ob $(T(\mathcal{A},\mathcal{B}))$ are DG-functors Stand$(\mathcal{A}) \to \mathcal{B}$.

aka $\boxed{A_\infty\text{-functors}}$ $\mathcal{A} \to \mathcal{B}$ ...

Stand($\mathcal{A}$) is monad, composition of two adjoint functors, maps to the stability ("twisting cochains" of Quillen)
- gives way to understand A_∞-functors.
& get DG model this way for $T(\mathcal{A}, \mathcal{B})$ ....

k field Example of comparison of Keller & Kontsevich models:

$T(A, k^{\text{pretr}})$ (k^{pretr} = complexes with f.d. cohomology) A = DG algebra for simplicity

Keller: semifree ~~DG~~ - A-modules M such that $\dim_k H^\cdot(M) < \infty$

Kontsevich: __finite__ complexes of __fin dim__ k-vector spaces with a $\boxed{\text{weak}}$ action of A.

Weak action: for any $a \in A$ have an endomorphism $f_a : C \to C$ (depending linearly on a), $f_{ab} \neq f_a f_b$ necessarily, but

$f_{ab} - f_a f_b = d(\varphi_{ab})$,
some combo of φ, f which is cocycle should be coboundary, etc...

- come naturally to this by trying to construct f.d. model of Keller module M. M is homotopy equivalent to a f.d. complex C, get for $a \in A$ an endomorphism up to homotopy, will satisfy A_∞ relations.

[Gain fin dim by losing strictness of action]

Quotients of DG Categories

Verdier: $\mathcal{T}$ triangulated, $Q \subset \mathcal{T}$ full triangulated subcategory
get $\mathcal{T} \longrightarrow \mathcal{T}/Q$ quotient triangulated category with universal property wrt such quotients.

What is DG version: $\mathcal{B} \overset{\text{full}}{\subset} \mathcal{A}$ DG categories [Keller]

2-categorical version advantage: works over rings
disadvantage: less precise (weaker form of uniqueness)

Theorem-definition A 2-categorical quotient of $\mathcal{A}$ and $\mathcal{B}$ is a pair $(\mathcal{C}, \xi)$ of a DG category $\mathcal{C}$ & a quasifunctor $\xi \in T(\mathcal{A}, \mathcal{C})$, s.t. the following equivalent properties hold:

$\mathcal{C} \in$ DGCat

1. $Ho(\xi) : Ho(\mathcal{A}) \to Ho(\mathcal{C})$ identifies $Ho(\mathcal{C})$ with $Ho(\mathcal{A}) / Ho(\mathcal{B})$ ... ie $Ho(\mathcal{A}) \longrightarrow Ho(\mathcal{C})$ ($\mathcal{B}$ killed in $Ho\,\mathcal{C}$ $\searrow Ho(\mathcal{A})/Ho(\mathcal{B})$ $\exists$/ factorization and resulting functor from Verdier quotient is an equivalence).

2. $\forall \mathcal{C}' \in$ DGCat the functor $T(\mathcal{C}, \mathcal{C}') \to Ker(T(\mathcal{A}, \mathcal{C}') \to T(\mathcal{B}, \mathcal{C}'))$ is an equivalence [when $\mathcal{B} \to \mathcal{A} \to \mathcal{C}'$ is zero]

(Verdier setting: universal property $Ker(T(\mathcal{A}, \mathcal{C}') \to T(\mathcal{B}, \mathcal{C}')) \simeq T(\mathcal{C}, \mathcal{C}')$ every functor killing $\mathcal{B}$ factors through $\mathcal{C}$)

Such $(\mathcal{C}, \xi)$ $\exists!$ in 2-categorical sense

Not obvious that either implies the other.
Recall $Ho(\mathcal{A}) = T(k^{pretr}, \mathcal{A})$ while second property formulated in terms of $T(\mathcal{A}, -)$

k field each $T(\ )$ is actually a DG-category $\Rightarrow$ more precise version, quotient unique in stronger sense.
Problem is not to prove but to formulate question when k is a ring...

Suppose $T(\mathcal{A}, \mathcal{C})$ comes from a DG category $DG(\mathcal{A}, \mathcal{C})$
– above formulation means have collection of objects $\xi_i \in DG(\mathcal{A}, \mathcal{C})$ together with a homotopy class of homotopy equivalences $\xi_i \to \xi_j$
– for more precise version need either specific ξ or rather contractible space of morphisms from the objects ξ_i, ξ_j (ie P resolution of k, $P \to Hom(\xi_i, \xi_j)$)
k not a field don't have $DG(\mathcal{A}, \mathcal{C})$ yet...

Def A DG quotient of $\mathcal{A}$ mod $\mathcal{B}$ is a diagram
$$\begin{tikzcd} \tilde{\mathcal{A}} \arrow[d, "\text{quasi-eq } S"'] \arrow[r] & \mathcal{C}, \\ \mathcal{A} \end{tikzcd}$$
-- in particular get object of $T(\tilde{\mathcal{A}}, \mathcal{C}) = T(\mathcal{A}, \mathcal{C})$.

s.t. $(\mathcal{C}, \xi)$ is a 2-categorical quotient.

$\longrightarrow$ uniqueness (and field) in ∞-category of DG-categories (ie DG-model of DG Cat)

... replace T by DG: $DG(\mathcal{C}, \mathcal{C}') \to \mathrm{Ker}(DG(\mathcal{A}, \mathcal{C}') \to DG(\mathcal{B}, \mathcal{C}'))$ is a quasiequivalence ~~-~~ automatic from definition!, just need to define $DG(\mathcal{C}, \mathcal{C}')$

---

2 constructions. I: new (Drinfeld), works if $\mathcal{A}$ is homotopy flat.

$\mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{A}$, add new morphisms.

$\forall X \in \mathcal{B} \subset \mathcal{A}$, add $\varepsilon_X : X \longrightarrow X$ of degree -1,

$d\varepsilon_X = id_X$. $\mathcal{C}$ is freely generated by $\mathcal{A}$ and these ε_X.

$\mathcal{A} \longrightarrow \mathcal{C}$.

If $\mathcal{A}$ is homotopically flat this gives desired answer

Rather than inverting quasi isomorphisms think of killing objects: ie killing all morphisms to & from this object (homotopically) — enough to kill identity morphism!

II. Keller's definition: consider inclusion $\underline{\mathcal{A}}$ = semifree $\mathcal{A}$-mod

$\underline{\mathcal{B}} \hookrightarrow \underline{\mathcal{A}}$ full subcategory: or have DG functor $M \longmapsto M \otimes_{\mathcal{B}} \mathcal{A}$ fully faithful on mod.

$\underline{\mathcal{A}} \supset \mathcal{B}^\perp = \{ M \in \underline{\mathcal{A}} : M|_{\mathcal{B}} \text{ is acyclic} \}$

$\iff \mathrm{Ext}^i(h_b, M) = 0 \quad \forall b \in \mathcal{B}$

h_b representable functor

Fact $Ho(\mathcal{B}^\perp) \xrightarrow{\sim} Ho(\underline{\mathcal{A}}) / Ho(\underline{\mathcal{B}})$ is an equivalence

$Ho(\mathcal{A}) / Ho(\mathcal{B})$ fully faithful.

So define quotient $\mathcal{C} = \{ M \in \mathcal{B}^{\perp} : \exists\, a \in \mathcal{A}$ s.t. $\exists$ dist triangle $a \to M \to N \to a$ with $N \in \underline{\mathcal{B}} \}$

-- essential image of $Ho(\mathcal{A}) / Ho(\mathcal{B})$

Here don't have DG functor $\mathcal{A} \to \mathcal{C}$ but need $\mathcal{X} \to \mathcal{C}$, $\downarrow$

... try to assign M to a, using a choice, but with contractible space of choices (preserving equivalence of course)..

equivalently construct N: $a \in \mathcal{A} \Rightarrow a \to M \to P \in \underline{\mathcal{B}}$, $P \xrightarrow{f} a$

f of deg 0 $(P = N[-1])$

so that $\mathrm{Cone}(f) \in \mathcal{B}^{\perp}$ ie $\mathrm{Cone}(f)|_{\mathcal{B}}$ is acyclic

ie $f|_{\mathcal{B}}$ is a quasi isomorphism

Restriction to $\mathcal{B}$: think of a via representable functor $h_a \in \mathcal{A}^{\circ}$-DG-mod

take $h_a|_{\mathcal{B}}$ and take a semifree resolution $P \to h_a|_{\mathcal{B}}$

— this is our choice, but unique in strongest homotopical sense (contractible space of choices).

Not self-dual definition ... so would like universal property to identify all.

How might you come to this def? why semifree DG-modules? if believe DG quotient exists: $\mathcal{A} \to \mathcal{C}$ with property!

Yoneda: $\mathcal{C} \hookrightarrow \mathcal{C}^{\circ}\text{-DG mod} \xrightarrow{\text{restrict}} \mathcal{A}^{\circ}\text{-DG-mod}$

$\Rightarrow Ho(\mathcal{C}) \to Ho(\mathcal{A}^{\circ}\text{-DG-mod}) \longrightarrow D(\mathcal{A}^{\circ}) = Ho(\underline{\mathcal{A}})$

- fully faithful, so natural to look for $\mathcal{C}$ in $\underline{\mathcal{A}}$.

In fact $\underline{\mathcal{A}}$ (infinite version) is implicit in Verdier:

$\mathrm{Ob}\, \mathcal{T}/\mathcal{Q} = \mathrm{Ob}\, \mathcal{T}$, Morphisms: invert $\mathcal{Q}$-quasi isomorphisms - ie morphisms with cone in $\mathcal{Q}$.

So morphisms are zigzags $\cdot \leftarrow \cdot \rightarrow \cdot \leftarrow \cdot \rightarrow \cdot$, $\swarrow \in \mathcal{Q}$-qism

but sufficient to take $\swarrow \searrow$ or $\searrow \swarrow$.

Verdier: $X, Y \in Ob\, T$ $\quad Ext^i_{T/\varphi}(X,Y) =$

$= \varinjlim_{C_Y} Ext^i_T(X, Y')$ over all "injective resolutions" $Y \xrightarrow{\text{quis}} Y'$;

C_Y = category with objects Q-quis $Y \to Y'$

ie maps $X \to Y'$ give map $X \dashrightarrow Y$ in derived category

& C_Y is a filtering category.

$\Rightarrow$ ... need for group structure on the limit of Ext's

filtering: commute with finite projective limits (eg products)

— Claim this is same infinity as $M \xrightarrow{} A$.

... can write via homotopy colimit, but Keller's construction takes care of this via semifreeness.

Exercise $a, a' \in A$, $P \xrightarrow{qis} A_a/_B$ semifree resolution

$P \in \underrightarrow{B}$. Show directly that

$$Ext^i_{Ho(A)/Ho(B)}(a', a) = \varinjlim H^i Hom(a', M) \quad \text{Verdier answer}$$

$M \in \{M, a \to M$ with cone in $Ho(B)\}$ up to homotopy

$\|$

Keller answer $\quad H^i(Hom_A(a', Cone(P \otimes_A B \to a)))$

- write $P = \bigcup_i P_i$ union of f.g. modules (filtering family)

-- ie cohomology of inductive limit over filtering set vs direct limit of cohomology, -- slightly different filtering categories...

Get functor from one to another, by cofinality of one sequence in another it is an equivalence.