~~NEO~~
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Some doable problems

• Important distinction between usual & geometric Langlands!
usual theory: can consider reducible & irred Gal reps separately $\Longleftrightarrow$ treat Eisenstein & cusp separately

Geometric story: they live together! reducible reps can be deformed to irreps!
Today: 2 separate stories: GKV construction of irred GL_n, Braverman-Gaitsgory for geometric Eisenstein, satisfactory for generic abelian L.S.

Problem 1 • Take abelian loc sys, deform it to be irreducible (consider universal deformation)
$\Rightarrow$ how to construct Langlands transform of this universal deformation? canonical formal deformation of geometric Eisenstein series as Hecke eigensheaf

— interesting even for Gl_2: can be done using ~~Drinf~~ F-G-V technique but would be better to develop in more uniform fashion

G reductive $\supset T$ max torus, ${}^LT$ dual torus

Given ${}^LT$ local system ρ on X, generic
$\Rightarrow$ Eis_ρ Eisenstein series (Braverman-Gaitsgory): perverse sheaf on Bun_G

$Eis_\rho = \bigoplus_\lambda Eis_\rho^\lambda$ graded object labelled by λ coweights of T

... version of infinite series.

Graded object $\Longleftrightarrow$ torus action: ${}^LT$ acts on Eis_ρ with characters λ.

ρ ${}^LT$ loc sys $\Rightarrow$ ρ considered as ${}^LG$ local system

Want: $\forall$ ${}^LG$ local system $\rho \Longrightarrow$ Langlands transform, a complex ${}^L\rho$ on Bun_G.

As ρ varies, fibers of ${}^L\rho$ (complexes of vector spaces) should form complex of $\mathcal{O}$-modules on stack of local systems.
Even for fixed ρ have nontrivial stack since ρ has automorphisms: grading is thus coherent sheaf on $\underline{B{}^LT}$ (autom of ρ generic are ${}^LT$)

Fix ρ_0 & consider formal stack S of ρ's infinitely close to ρ_0: quotient of formal variety by a group (namely ${}^LT$):
can write $S \simeq V/{}^LT$.

$V = \mathrm{Spec}\ \mathcal{O}$ complete local ring, defs of completely reducible local systems $\Longrightarrow V$ singular
GL_2 case $\Rightarrow$ quadratic cone singularity,
generally systems of quadratic eqns.
Think of π_1 standardly $\prod [A_i, B_i] = 1$,
get quad. relations from defs of abelian reps

Problem: Construct ${}^L\rho$, an $(\mathcal{O}, \mathcal{D})$-module on $S \times Bun_G$
l-adic setting can also specify class of objects:
perverse sheaf of $\mathcal{O}$-modules with ${}^LT$ action

Restrict to $B{}^LT \subset S$ should get Eisenstein series, & ${}^L\rho$ should be Hecke eigenvector.

Technique to solve $\dashrightarrow$ semiinfinite flags II
Feigin-Finkelberg-Mirkovic : geometric realization of ${}^L\mathfrak{g}$ ... get e.g. ${}^L\mathfrak{g} \circlearrowright \mathrm{Eis}_{\text{partial}}$

Hint: by deformation theory of local systems: controlled by cohomology of Lie algebra twisted by loc sys.
$H^0 \longleftrightarrow$ symmetries $\quad H^1 \longleftrightarrow$ deformations

$\infty/2$ Flags II $\Rightarrow$ ${}^L\mathfrak{g} = H^0(\ ) \longrightarrow \mathrm{End}\,\mathrm{Eis}_{triv}$

Now want some kind of dgla's $\Rightarrow$ solve problem in char. 0 of coefficients...

To get away from char 0 need not dgla versions but divided powers versions: not $U\,{}^L\mathfrak{g}$ needed but distributions on group should act

Char. 0: have dgla $R\Gamma(X, {}^L\mathfrak{g}_{\rho_0})$

Want to construct map $\downarrow$ $\qquad$ deformation theory $\Rightarrow$ abstract nonsense

$R\mathrm{Hom}(\mathrm{Eis}_{\rho_0}, \mathrm{Eis}_{\rho_0})$,

should be isomorphism: Rankin-Selberg (Lysenko) suggests this is an isomorphism...

If everything is abelian (eg Eis all perverse) $\Rightarrow$ probably don't need homotopical algebra of dgla's, with higher data...

Q: What happens when ρ_0 = trivial?

<u>GL_2 picture:</u> (de Rham setting for linguistic convenience)

Assume genus big enough (high codim of difficult local systems)

<u>Naive hope:</u> $(\mathcal{O}, \mathcal{D})$-module on $\mathrm{LocSys}_{GL_2} \times \mathrm{Bun}_{GL_2}$

flat / LocSys & Hecke eigensheaf $\qquad (\mathcal{O} \qquad \mathcal{D})$

... <u>can</u> realize this naive hope on open substack $\mathrm{LocSys}^{easy}_{GL_2} \times \mathrm{Bun}_{GL_2} \Rightarrow (\mathcal{O},\mathcal{D})$-module M^{easy}

ρ <u>difficult</u> if reducible & irred components isomorphic to each other -- eg trivial not difficult!

<u>Theorem</u> M^{easy} uniquely extends to an $(\mathcal{O},\mathcal{D})$-module M on $\mathrm{LocSys}_{GL_2} \times \mathrm{Bun}_{GL_2}$ so that M is Cohen-Macaulay over LocSys_{GL_2}, Hecke eigensheaf.

Replace flatness by CM, weaker property, <u>not flat</u>

(LocSys are l.c.i, in particular CM, so no contradiction flat vs CM!

On a smooth variety being flat $\Leftrightarrow$ CM, general singular flat $\not\Rightarrow$ CM

Specialize to trivial loc sys get infinite (sketched above) complex of $\mathcal{D}$-modules
Proof very technical & computational.

Q: Does M define an equivalence of categories?
A: Which categories? bdd or not, etc?

Analogous situation: Fourier transform b/w of fns on H abelian group & dual $H^\vee$ ... which class of functions? need to specify.
Geometric rep theory: fns $\longrightarrow$ complexes.
function spaces $\longrightarrow$ derived categories
Need geometric functional analysis, consider varying classes of functions
test situations: eg local systems with tame ramification at some points & fixed local monodromies
if at least one local monodromy nontrivial $\Rightarrow$ easy local systems. Need to think about geom functional analysis: stack Bun_G in this case not quasicompact, loc sys singular so have different notions of boundedness / perfection

Arinkin: check equivalence of derived categories in simplest truly nonabelian situation, on $\mathbb{P}^1$ with ramifications to eliminate all reducible loc sys, remove all singularities / infinities, secure genericity.

To construct ^{L}(triv): could try to use Eisenstein series construction... grading $\longleftrightarrow$ torus symmetry.
But for triv rank 2 loc sys, symmetry is $GL_2 \Rightarrow$ need sheaf on BGL_2 not BT

F-dim rep of GL_2 decomposes under ^LT, labeled by pairs of integers & $V^{m,n} = V^{n,m}$ on wt spaces (Weyl symmetry)
– so if geom Eisenstein series carried GL_2 action

would have such naive functional eqn $\longrightarrow$
on functions would get naive fn. eqn
as if Eis had no poles.

Suppose $f(t) = \sum a_n t^n$ $\quad g(t) = \sum b_n t^n$
formal ~~series~~ Laurent in two variables in ~~opposite direction~~

What does $f(t) = g(t^{-1})$?

naively means finite series, Laurent polynomials.
More sophisticatedly: two Laurent expansions
of mero function,

$a_n - b_n$ is linear combo of geometric
progressions (δ-fns) $\longleftrightarrow$ pole of
Eis series when two loc sys are equal,
so no naive equality $V^{n,m} = V^{m,n}$.

So don't have action of GL_2 since Eis do have
poles!

Mirkovic-Finkelberg etc: Eis ~~riv~~ carries action of the Lie algebra
$\log$ -- so don't get sheaf on stack BGL_2.
Maybe this is wrong object.

Fourier transform: $\hat{f}(y) = \int f(x)\, K(x,y)\, dx$
inverse Fourier: put in minus signs
Geometric version: replace kernel sheaf by
Verdier dual.

Orthogonality: $\int e^{ix\xi} e^{-ix\eta}\, dx = \delta(\xi - \eta)$

- need such both ways, one
over Bun_G in D-mod sense & one
over LocSys " O-mod "

Orthogonality / D-mods: Lysenko (on Bun_G):
analog of computation of norm squared of normalized
Hecke eigenfunctions - GL_n answer by
Rankin-Selberg method $\Rightarrow$ geometric analog for GL_n.

Orthogonality over Loc Sys: no standard tools to compute.
Arinkin: in simplest nontrivial case really do get δ-fn argument... mysterious proof however.

Lysenko: only irred loc sys: $\mathcal{D}$-mods corresponding to two orthogonal loc sys are orthogonal... scalar square

~~Fourier on R restricted to open subset~~

2. An important technical tool – Another compactification of Bun_B, or Spaces of rational maps ⇢ different from Drinfeld quasi-maps

$G \supset B$

$$\begin{tikzcd}
Bun_B \arrow[r, hook] \arrow[drr, "\pi"'] & & \overline{Bun}_B \arrow[d] \arrow[dd, bend left=60, "\bar\pi"] \\
& & \underline{Bun}_B \arrow[d] \\
& & Bun_G
\end{tikzcd}$$

Construct relative compactification of this representable morphism:

$\pi^{-1}(\mathcal{F})$ = B-structures on $\mathcal{F}$ = sections of $(G/B)_{\mathcal{F}}$ = $\Gamma(X,(G/B)_{\mathcal{F}})$

$\bar\pi^{-1}(\mathcal{F}) = \Gamma_{quasi}(X,(G/B)_{\mathcal{F}})$: Drinfeld compactification
quasi-sections

Fibers of $\underline{Bun}_B \to Bun_G$ will be compact but not algebraic varieties -- $\underline{Bun}_B$ not algebraic stack
just spaces as fibers

Spaces $\underline{Bun}_B(S) = \{ \mathcal{F}$ G-bundle , rational section of $(G/B)_{\mathcal{F}} \}$

rational section = section on $X \times S - D$, D finite$/S$

eg X $\mathcal{F}$ = triv $\Rightarrow$ all rational functions to G/B
– any G-bundle $\mathcal{F}$ becomes trivial on nonempty open subset, so just get rational maps to G/B...

$\Rightarrow$ study topological object $Rat(X,Y) = Rat_{X\to Y}$ Y alg variety

$\mathrm{Rat}_{X\to Y}(S)$ = set of rational maps $X \times S \longrightarrow Y$
$= \varinjlim_{\substack{D \subset X\times S \\ \text{finite}/S}} \mathrm{Mor}(X \times S \setminus D \longrightarrow Y)$

(require D ~~nowhere~~ dense in fiber

Suppose ground field $= \mathbb{C}$
$S/\mathbb{C}$ $\Rightarrow$ S-point φ of $\mathrm{Rat}_{X\to Y}$ defines map
$\varphi_*: S(\mathbb{C}) \longrightarrow \mathrm{Rat}_{X\to Y}(\mathbb{C})$

want all such φ_* to be continuous –
put strongest topology s.t. all φ_* are continuous.

<u>Example</u> $\mathrm{Rat}(\mathbb{P}^1, \mathbb{P}^1)$: most naively
rational maps from ~~complete~~ $\mathbb{P}^1$ to $\mathbb{P}^1$ is regular
$\Rightarrow$ look at regular maps, split as disjoint
union of maps of fixed degree....
here we'll only get <u>filtration</u> by degree:

$\mathrm{Rat}_{\leq 1}(\mathbb{P}^1, \mathbb{P}^1)$: $z \longmapsto \frac{az+b}{cz+d}$
$\leftrightarrow$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ up to scalar, nonzero
$\Rightarrow$ $(a:b:c:d) \in \mathbb{P}^3$ If matrix is degenerate
$ad = bc$ get degree 0 map
– quadric in $\mathbb{P}^3$ $Q: \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$
$ad=bc$ $(a:b:c:d) \longmapsto \{\frac{a}{c} = \frac{b}{d}\}$ $\mathbb{P}^1$.
$\{a/b = c/d\}$ $\mathbb{P}^1$

(constant map $z \mapsto \frac{a}{c} = \frac{b}{d}$
So projection on this $\mathbb{P}^1$
is just the map we want

$\mathrm{Rat}_{\leq 1}$ = take $\mathbb{P}^3$, & contract quadric Q
to $\mathbb{P}^1$ under map $\frac{a}{c} = \frac{b}{d}$.
– forbidden in algebraic geometry, but
fine as topological space.

$\overline{Bun}_B \times_{\underline{Bun}_B} \overline{Bun}_B$ equivalence relation on $\overline{Bun}_B$ which is (rel) proper (two projections are proper)

$$\begin{array}{c} \overline{Bun}_B \\ \downarrow \\ \underline{Bun}_B \end{array}$$

- restrict direct images,

e.g. GL_2: rk 2 vector bundle

$L,\ A_1, A_2 \subset L$ s.t. $A_1 \wedge A_2 = 0$

Act map $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is ratio of two homog polynomials, fix degree $(f:g) \in \mathbb{P}^n$

$\rightsquigarrow$ linear subspace in projective space for equiv relation $pf = qg$.

~~Perverse~~ sheaves etc: can define on alg variety with proper equivalence relation — get derived category of sheaves, with Verdier duality etc.

— study sheaves upstairs with descent data

- two notions of equivariance ! & *

$R \subset X \times X$ proper equivalence relation, $R \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{\pi} \text{"}X/R\text{"}$

Def'n: $\left\{ \begin{array}{l} \text{complex } \mathcal{F} \text{ on } X \text{ with} \\ p_1^* \mathcal{F} \xrightarrow{\sim} p_2^* \mathcal{F} \end{array} \right\} = \mathcal{F}$ think of as complex on quotient X/R

Problem: define what is $\pi^! \mathcal{F}$?
& ! descent datum on $\pi^! \mathcal{F}$.

Suppose X/R exists in usual sense — use proper base change

$$\begin{array}{ccc} R & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & X/R \end{array} \quad \leftarrow \mathcal{F}$$

Compute $p_1^* \pi^! \mathcal{F}$ by b.c.

Adjunction $\mathcal{F} \longrightarrow \pi_* \pi^* \mathcal{F} = \pi_* \mathcal{F}$

$\mathcal{F} =$ "Ker" $(\pi_* \mathcal{F} \longrightarrow$

$X \times_Y X \times_Y X$
$X \times_Y X$

Suppose X/R exists, know $\pi^* \mathcal{F} = \mathcal{F}$. Now write simplicial scheme, use proper base change for $\mathcal{F} \to \underbrace{\pi_* \pi^! \mathcal{F}}_{\underset{\mathcal{F}}{\shortparallel}} \to \cdots$

$\Rightarrow$ resolve $\mathcal{F}$ in terms of direct images of things given in terms of $\mathcal{F}$

Want to describe $\pi^! \mathcal{F}$ from $\mathcal{F}$ & descent datum in case everything exists!

resolve $\mathcal{F}$ using $\mathcal{F}$'s, but by base change know ! - pullback of $*$-direct image

So if $X/\mathbb{R}$ doesn't exist still have this complex on the tower -- want a simplicial scheme

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Geometric Eisenstein series

- take "intermediate pullback" of perverse sheaf on Bun_T to $\overline{Bun}_B$, & pushforward to Bun_G

$$\begin{tikzcd} \overline{Bun}_B \arrow[r] \arrow[d] & Bun_T \\ Bun_B \arrow[ur] \arrow[d] & \\ Bun_G & \end{tikzcd}$$

Proposal: consider pushforward to $\underline{Bun}_B$ before to Bun_G

"Non proved theorem": fibers of $\underline{Bun}_B \to Bun_G$ are contractible

Another potential application:

$G = GL_2$: $\quad \underset{\substack{\shortparallel \\ \text{rk 2 + fixed rk 1 subbundle}}}{Bun_B} \to Bun_G = \text{rk 2 bundle}$

Can construct complex on Bun_B assoc to inved local system, now need to descend it to Bun_G.

Use fact that fibers are projective space - hypersurface:

have ~~to~~ $\overset{\text{complex}}{\text{costs}}$ sheaf on Bun_B with no restriction at ∞, need perversity to descend

(descent for complexes were complicated)

- need contractibility

If don't impose irred local systems would probably get nonperverse complexes — would be nice to have something with contractible fibers over Bun_G related to Bun_B ....

General contractibility statement: [Contractibility $\Leftrightarrow$ π_1 vanishes & cohomology with trivial coefficients vanishes — both make sense in this general context.]

Let X = irreducible variety (eg curve) $\quad E = k(X)$
Y = scheme of finite type $/E$
$\Rightarrow$ associate fake space (functor)
Y_E : k-schemes $\longrightarrow$ sets
-- convenient to choose a model for Y as generic fiber of $\mathcal{Y} \longrightarrow X$ scheme of finite type

$$Y_E(S) = S\text{-families of rational sections of } \mathcal{Y} \to X$$
$$= \varinjlim_{D} Mor_X\left((X\times S)\setminus D, \mathcal{Y}\right)$$
fibers of D nowhere dense in X [?]

-- independent of model $\mathcal{Y}$.

<u>To be checked</u>: Suppose Y is irreducible & $\forall y \in Y$ $\exists$ $y \in U \subset Y$ Zariski nbhd isomorphic to an open subset of $\mathbb{A}^n$ (stronger than rationality)
Then Y_E is contractible.
(direct limit of π_1 & H^i is zero..)

(note if $Y = \bigcup_i V_i \Rightarrow Y_E = \bigcup_i (V_i)_E$ hopefully still open!

- case with spaces of rational maps to affine spaces
$U \subset \mathbb{A}^n$ nonempty $\Rightarrow$ claim U_E is contractible

3. Microlocalization

deRham setting : $\mathcal{E}$-modules

[Parallel picture for perverse sheaves (Kashiwara … not suitable to complex algebraic context … requires non-constructible sheaves)
& modification by MacPherson - Vilonen - S. Gelfand]

$T^*M - 0$
$\pi \downarrow G_m$
$\mathbb{P}(TM)$

M smooth algebraic variety : $\mathbb{P}(TM) := (T^*M \setminus 0)/G_m$
contact variety, with sheaf of associative filtered algebras $\mathcal{E}$, complete wrt filtration

$$gr\, \mathcal{E} = \pi_* \mathcal{O}$$

$\pi_* \mathcal{O}$ is $\mathbb{Z}$-graded, with Poisson bracket of degree -1

Define $\mathcal{E}$: sections over open affine subset (assume M affine — construction local in M, $M \times \mathbb{P}(-)$ product)
f homog fn on T^*M (look at complement $\mathbb{P}(T^*M) \setminus \{f=0\}$) $=: \mathbb{D}(f)$

$$\Gamma(\mathbb{D}(f), \mathcal{E}) = \text{completion wrt filtration} \left(\mathcal{D}_M \text{ ring}, \text{ invert all diffops with principal symbol } f \right)$$

Suppose we've inverted diffop $F \Rightarrow$
can invert (in completion) operators of form $F + G$ where G is of lower degree

Equivalent def locally : write diffops via symbols
$\sum a_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha \rightsquigarrow \sum a_\alpha(x) \xi^\alpha$
Symbol of a product expressed via symbols of factors via formula which makes sense for symbols more general fns than polynomials! $f(x,\xi)$ infinite sums of homog functions in x, ξ. $\rightarrow$ explicit construct…

Let $p: \mathbb{P}(TM) \to M$, natural map $p^{-1}\mathcal{D} \to \mathcal{E}$
$\quad\;\; \mathcal{E} \qquad\quad\;\; \mathcal{D}$

N $\mathcal{D}_M$-module $\Rightarrow$ $\mathcal{E} \otimes_{p^{-1}\mathcal{D}_M} p^{-1}N$ microlocalization of N.

Would like to apply to $M = Bun_G X$ --- locally quotient of smooth variety by a group... assume we understand notion of $\mathcal{E}$-module on $Bun_G X$.

$T^* Bun_G = \{ (\mathcal{F}, \eta) : \mathcal{F}$ G-bundle on X, $\eta \in H^0(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X) \}$ Higgs bundles

Def η is weakly generic if $\eta(x)$ is regular for $x \in X$ generic. (regular elts invariant under $G \times \mathbb{C}^*$)

Problem Given a local ${}^LG$-system ρ on X, compute/define/construct the restriction of the microlocalization of ${}^L\rho$ to $\mathbb{P}(T^* Bun_G)$ weakly generic

Hecke correspondences on $T^* Bun_G$ preserve Hitchin fibration, & more! if $(\mathcal{F}, \eta)$ corresponds to $(\mathcal{F}', \eta')$ $\mathcal{F} \cong \mathcal{F}'$ identified generically & η, η' agree there.

$X \longrightarrow Y$ map of smooth varieties $\rightarrow$ Lagrangian submanifold in $T^*X \times T^*Y$.
Correspondence $Z \subset X \times Y$ $\Rightarrow$ Conormal of $Z \subset T^*(X \times Y)$
X, Y smooth

$\Rightarrow$ Lagrangian correspondences between cotangent bundles.

So for Hecke correspondences: if $(\mathcal{F}, \eta)$ $(\mathcal{F}', \eta')$ correspond $\Rightarrow$ $\mathcal{F} \cong \mathcal{F}'$ generically sending η to η'.
So if one is generic the other is
So can consider Hecke eigensheaves restricted to generic part
$\because$ Hecke operators preserve them.

"Theorem" Sing Supp (${}^L\rho$) $\overset{\text{as cycles}}{=}$ zero fiber of Hitchin fibration
$T^* \mathrm{Bun}_G \longrightarrow$ ~~vector~~ $\mathbb{C}^*$ space of half the dimension
with unique fixed point
0-fiber is well-defined cycle
... Laumon proved this for GL_2 (GL_n?)
for ρ ~~on~~ oper, this property clear by construction.

Zero fiber: Higgs field η is nilpotent everywhere.
(at least set theoretically ... careful with multiplicities)
... global nilpotent cone

We're looking only at η which is regular nilpotent generically

"Kashiwara Theorem": $L \subset \mathbb{P}(T^*M)$ Legendrian & smooth
{Describe ~~of~~ $\mathcal{E}$-modules set-theoretically supported on L}
"="
{local systems on L}

$\mathcal{E}$-modules here all fin. generated with Legendrian support... "holonomic"

.. careful: need to replace $\mathcal{D}$-modules by $\omega^{\frac{1}{2}}$ twisted $\mathcal{D}$-modules & correspondingly change $\mathcal{E}$.
Similarly should consider $\frac{1}{2}$canonically-twisted local systems.

So we're in generic part of global nilpotent cone, describe Hecke $\mathcal{E}$-modules as local systems on smooth Legendrian ... restrict to smooth locus

Conjectural (partial) answer for $G = GL_2$

Higgs bundle: $(L \text{ rank } 2,\ \eta: L \longrightarrow L \otimes \omega_X)$
nilpotent: $\eta^2 = 0$, η is not identically zero

When is (L, η) a smooth point of the nilpotent cone?
(as reduced scheme)

Answer: iff all the zeros of η are simple.
Number of zeros can be arbitrary & labels the irreducible component for which (L,η) is generic. -- ∞ many components (non quasicompact stack).

Suppose η has k simple zeros.
Conjecture: Fiber of our local system at (L,η) (coming from microlocalization of φ) is
$\rho_{x_1} \otimes \ldots \rho_{x_k}$

Construction: ρ irreducible construction of Langlands transform: achieve as Radon transform (Fourier for homogeneous functions):

Construct complex on projective space of Hom's $\leftrightarrow$ effective divisors on curve dual to $\mathbb{P}\,Ext(L_1, L_2)$ on moduli of B-bundles.
Apply Radon to $Sym^k \rho$ get perverse sheaf on Bun_B.

Radon transform on singular supports:
$\mathbb{P}(T\mathbb{P}^n) = \{ x \in \mathbb{P}^n \,\&\, H \ni x$ hyperplane $\}$
self-dual variety --- same for $\mathbb{P}(T\check{\mathbb{P}}^{n})^*)$
So $\mathbb{P}(T\mathbb{P}^n) = \mathbb{P}(T\check{\mathbb{P}}^n)$ dual projective space
& ~~Radon~~ $(SSupp(N)) \xrightarrow[Radon]{} SSupp(Radon\ N)$
under this identification
So Laumon calculates SSupp of Gl_n Langlands sheaves using this construction:

In fact full microlocalization behaves well under Radon $\Rightarrow$ can compute microlocalization of Gl_n Langlands sheaves.

$\mathfrak{g}^* = \mathfrak{g} \supset \mathfrak{n}$

η reg nilpotent $\Rightarrow$ flag of $\mathcal{F}$ generically
$\Rightarrow$ flag everywhere, preserved by
$\eta \in \Gamma(X, \mathfrak{n}_{\mathcal{F}} \otimes \omega_X)$

$\mathfrak{n}_{\mathcal{F}}$ has canonical filtration (central series)
$gr\ \eta \in (\bar{\mathfrak{n}} = \mathfrak{n}/[\mathfrak{n},\mathfrak{n}])_{\mathcal{F}} \otimes \omega_X$
$\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\oplus \mathbb{C}_v}$ v vertex of Dynkin diagram

So get invariants of η which are vertices of Dynkin diagram -- divisor labelled by such.
So can formulate generalization of Gl_2 conjecture to all G..., despite lack of direct construction!

Nice feature of this picture: answer is local, involves only points where η not regular
... might expect such factorization even outside of smooth locus - factorization for maps from curve to nilpotent cone, labelled by fundamental weights ?? And it's come up actually for other groups in constructing this sheaf on Ssupp from local systems ρ.

Answer manifestly independent of choices such as passage to Bun_B..

Gl_2: fiber of Langlands transform of generic points has rank 2^{3g-3}, can attempt to write as tensor product of $3g-3$ 2 dim vector spaces...

- but this is along 0 section, nongeneric part of nilpotent cone!
Claim is that this picture literally holds on generic part of nilpotent cone!

$\rightsquigarrow$ Should be related to Whittaker functions.
Classical automorphic forms for Gl_n! have multiplicity one.
No such for general G in classical automorphic theory

So this suggests a weakly generic locus have some multiplicity one results...

What's the relation to Whittaker sheaves?

(Dennis) Conjecture: consider sheaves on Bun_G with Whittaker coeffs $=0$ — does it coincide with D-mods whose singular support misses weakly generic locus?

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4. Motivic Bewilderment

$\{$Voevodsky Triangulated category of k-motives$\} \longrightarrow \{$Derived category of $Gal\ \bar{k}/k$-modules$\}$

ρ an irred local system on X, rank n

$\Rightarrow {}^L\rho$ on $Bun_{GL(n)}$

$\mathcal{F} \in Bun_{GL(n)}(k) \Longrightarrow ({}^L\rho)_{\mathcal{F}} \in \{$Derived category of $Gal\ \bar{k}/k$-modules$\}$

Q: If ρ is "motivic" can one define

$({}^L\rho)_{\mathcal{F}} \in \{$Triangulated category of motives$\}$?

— motivic automorphic forms??

example of motivic local system: Tate module of abelian variety over base, almost everywhere of good reduction.

More correctly: look at ρ of Artin type (finite image representations)

Simplest case: $n=2$. $\mathcal{F}$ rank 2 vector bundle on X. Define complex of Galois modules: first choose sufficiently negative rank 1 subbundle A of $\mathcal{F}$ $\Rightarrow$ ℓ-adic cplx of Galois rep, clearly of motivic nature

$({}^L\rho)_{\mathcal{F},A} \in$ Triang category of motives

— need to show independent of choice: for different A.

need issues of comparing motivic complexes

- our complexes defined using standard functors from ρ so are motivic, but arguments to show independent of A is not motivic in nature — uses fact that local systems on $\mathbb{P}^n/k = \bar{\mathbb{F}}$ is constant ... need motivic version of this (basic case is $\mathbb{P}^1$) — is any motivic local system on $\mathbb{P}^1$ constant?

Beilinson: Any polarizable VHS on constant local system is constant
For compact varieties
case of $\mathbb{P}^1$: constant local system $\Rightarrow$ Hodge filtration is a moving flag, it & complex conjugate give full space $\Rightarrow$ map $\mathbb{P}^1 \longrightarrow$ upper half plane ... which must be constant.

$({}^L\rho)_{F,A}$ Structure: $\tilde{X} \xrightarrow[G]{\pi} X$ $\quad \rho: G \to GL_n \bar{\mathbb{Q}}$

$Y_{F,A}$ a variety with action of $S_n \ltimes G^n$
(symmetric product of $\tilde{X}$)

$$({}^L\rho)_{F,A} = R\Gamma(Y_{F,A}, \bar{\mathbb{Q}}_\ell)^{\rho(n)} \text{ - isotypic component}$$

where ρ gives rise to $\rho^{(n)} := \rho \boxtimes \dots \boxtimes \rho$ rep of $S_n \ltimes G^n$

Motivic cohomology ... replace $R\Gamma(-, \bar{\mathbb{Q}}_\ell)$ by motive of the variety - object of additive ~~ab.~~ category, G acts on $Y \Rightarrow$ acts on its motives, get direct summands corresponding to isotypic components. So we have motives associated to F, A
... to show independence of A use description of local systems on $\mathbb{P}^n$, doesn't help us motivically

Classical version of Deligne's theorem: Artin L function for Artin rep has no pole (polynomial not Laurent polynomial) (Artin's conjecture)

Motivic bewilderment 2

Interpolation of Motivic Artin conjecture:
show certain cycles rationally equivalent to zero, for nontrivial ~~reasons~~...

... $\boxed{\text{Motivic Deligne Theorem}}$?

(Artin L-fn of function field is holomorphic)

smooth, proj X curve over some field

... comes from rep of finite quotient of π_1

Deligne's Theorem: ρ Artin type local system on X, geometrically irreducible, coeffs field of char 0 of rank $d > 1$. $\rho^{(n)}$ symmetric power of ρ: sheaf on n^{th} symmetric power $\mathrm{Sym}^n X$

$$\mathrm{Sym}^n X \xrightarrow{\pi} \mathrm{Pic}^n X \ ,$$

$$\boxed{\text{Deligne: For } n > d(2g-2), \ R\pi_* \rho^{(n)} = 0}$$

Artin: a certain function is holomorphic (polynomial)
almost all coeffs zero ... all coeffs greater than $d(2g-2)$ are zero!
Deligne theorem says most "coefficients" are zero

$\alpha \in \mathrm{Pic}^n X(k)$. Fiber $\left(R\pi_* \rho^{(n)}\right)_\alpha = R\Gamma(\text{fiber}_{\pi^{-1}(\alpha)}, \rho^{(n)}|_{\pi^{-1}(\alpha)})$

$\Rightarrow$ reformulate on level of fibers: ρ comes from finite étale G-cover of X $\quad \tilde{X} \xrightarrow{G} X$
$\rho : G \longrightarrow GL(V)$ $\dim V = d > 1$, irreducible

$$\tilde{X}^n \longrightarrow X^n \longrightarrow \mathrm{Sym}^n X \longrightarrow \mathrm{Pic}^n X$$

$\tilde{X}^n(k) \supset (\tilde{X}^n)_\alpha \xrightarrow{\text{fiber over } \alpha} \alpha \in \mathrm{Pic}^n(X)(k)$

Smooth for sufficiently generic α (in char 0 for general reasons, here for all characteristics)

$$\left(R\pi_* \rho^{(n)}\right)_\alpha = H^*((\tilde{X}^n)_\alpha, \mathbb{Q}_\ell)^\sigma$$

σ-isotypic component

$$S_n \ltimes G^n \circlearrowright (\tilde{X}^n)_\alpha$$
$$\circlearrowright V^{\otimes n}$$
$$\sigma :=$$

Deligne ... this σ-isotypic component is zero

What is motivic version? to rep σ of $S_n \ltimes G^n$ there corresponds an idempotent in the group algebra, $e_\sigma \in \overline{\mathbb{Q}}[H = S_n \ltimes G^n]$

→ [Assume $(\tilde{X}^n)_\alpha$ smooth]

Deligne: e_σ acts trivially on $H^*(\tilde{X}^n_\alpha, \mathbb{Q}_\ell)$

- each element in group algebra gives cycle in $Y \times Y$ whenever group G acts on variety Y.
- so e_σ gives cycle $E_\sigma \in \tilde{X}^n_\alpha \times \tilde{X}^n_\alpha$

Deligne's theorem says E_σ is homologically equivalent to zero.

Q: Is E_σ rationally equivalent to zero?
⟹ would imply acts trivially on motives...
- should follow from standard conjectures, but not clear geometrically!

Q) If Voevodsky motive has l-adic realization $= 0$ then motive should be zero?
- ... follows from some general conjectures & implies above.