M. Finkelberg : Toda Lattice & Quasimaps

$$QH^G(G/B) \overset{\text{heuristically}}{=} HF^G(\widetilde{\text{Loops in}}\ G/B)$$

equivariant quantum cohomology → universal cover

= equivariant T. cohomology of universal cover to LG/B

... theorem in differential geometric setting

RHS = Floer cohomology.

We'll define LHS in terms of RHS, now consider RHS.

& try to explain appearance of Toda lattice in Q-cohomology of flags.

$D = \operatorname{Spec} \mathcal{O}$, $\quad \mathring{D} = \operatorname{Spec} K$

$G(\mathcal{O}) \curvearrowright \operatorname{Maps}(\mathring{D}, G/N)/T$ ~~maps into base affine space~~

Orbits $\longleftrightarrow$ lattice $\check{\Lambda}_G$ of coweights.

0-orbit $\operatorname{Maps}^0 = \operatorname{Map}(D, G/N)/T$

$$\left.\begin{array}{l}\overline{\operatorname{Maps}^0} \subset \operatorname{Map}(D, \overline{G/N})/T \\ \text{maps st. } \mathring{D} \text{ lands in } G/N\end{array}\right\} \text{honest scheme of infinite type}$$

Orbits labelled by smaller coroots: orbit closure will be isomorphic, but lying inside.
Define periodic system of embeddings, gives ind-scheme structure for $\operatorname{Maps}(\mathring{D}, G/N)/T$
- each orbit is of infinite type

Convolution action of Gr_G on $\operatorname{Maps}(\mathring{D}, G/N)/T$
(ie $G(K)$ acts on loops, can convolve sheaves,)

$\Rightarrow$ action of $H_\bullet^{G(\mathcal{O})}(Gr_G)$ on $H_\bullet^{G(\mathcal{O})}(\operatorname{Maps}(\mathring{D}, G/N)/T)$

equivariant homology of Grassmannian acts on equivariant (A-Kapranov) homology of map space.

Arkhipov-Kapranov

$H_\bullet^{G(\mathcal{O})}(Gr_G)$ is a commutative ring,

$H_\bullet^{G(\mathcal{O}) \rtimes \mathbb{C}^*}(Gr_G)$ is $\underline{\text{noncommutative}} \Rightarrow$ Poisson structure on $H_\bullet^{G(\mathcal{O})}(Gr_G)$

$H_\cdot^{G(\mathcal{O})}(Gr_G) \supset H_\cdot^{G(\mathcal{O})}(pt)$ maximal Poisson commutative subalgebra: <u>integrable system</u>

$H_\cdot^{G(\mathcal{O})}(Gr_G) = \mathbb{C}[\mathfrak{z}^{G^\vee}_{\mathfrak{g}^\vee}]$

$H_\cdot^{G(\mathcal{O})}(pt) = \mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[\mathfrak{t}^\vee]^W$

$\mathfrak{z}^{G^\vee}_{\mathfrak{g}^\vee} \hookrightarrow \{(\underset{\in\mathfrak{g}^\vee}{X}, \underset{\in G^\vee}{g}) : Ad_g X = X,\ X \in \mathfrak{g}^{\vee\, reg}\} / Ad_{G^\vee}$

$= \mathbb{C}[\check{T} \times \mathfrak{t}^\vee, \frac{\alpha^\vee - 1}{\alpha^\vee}\quad \alpha \in R^+]^W$ $\qquad \alpha^\vee - 1 \in \mathbb{C}[T^\vee]$, $\alpha^\vee \in \mathbb{C}[\mathfrak{t}^\vee]$

$=$ Hamiltonian reduction $\mathbb{C}[T^* G^\vee // G^\vee]$

Kostant: $T^* G^\vee // G^\vee \simeq T^* G^\vee \underset{reg\ \psi\times\psi}{//} \check{N} \times \check{N}$

reduction at regular character via action of nilpotent subgroup by right & left subgroup

Functions $\mathbb{C}[T^* G^\vee \underset{reg}{//} \check{N} \times \check{N}]$ - sits inside localization: functions on big Bruhat cell $\check{N} w_0 \check{T} \check{N} = G^\vee_{w_0}$

$\mathbb{C}[T^* G^\vee_{w_0} // \check{N} \times \check{N}, \psi \times \psi] \simeq \mathbb{C}[T^* T^\vee]$

<u>Claim</u> This localization $\mathbb{C}[T^* T^\vee]$ is the Arkhipov-Kapranov $H^{G(\mathcal{O})}(Maps(\mathring{D}, G/N)/T)$

$\mathbb{C}[T^* T^\vee] \simeq \mathbb{C}[T^\vee \times \mathfrak{t}^\vee, (\frac{\alpha^\vee - 1}{\alpha^\vee})]^W$

slightly mysterious (commutative subalgebra of graded Hecke ..)

$$H^{G(O)}(pt) \subset H^{G(O)}(\mathrm{Maps})$$
$$\text{\rotatebox{90}{$\cong$}} \qquad\qquad \| $$
$$\mathbb{C}[\mathfrak{t}^\vee/W] \subset \mathbb{C}[T^\vee \times \mathfrak{t}^\vee]$$

classical exactly Toda lattice Hamiltonians

No product on Floer cohomology, but there is one on Q Coho, & Floer is free rank 1 module over Fun.

Similarly $K^{G(O)}(Gr_G) \supset K^{G(O)}(pt) = \mathbb{C}[T]^W$
close to, only equal for E_8

(universal centralizer variety) $\mathbb{C}[Z^{G^\vee}_{G^\vee}]$

$$K^{G(O)}(Gr_G) = \mathbb{C}\left[T^\vee \times T, \frac{\alpha^\vee - 1}{\alpha^\vee - 1}\right]^W$$

first factor
second factor

Localize this: invert

$$K_{\phi}^{G(O)}(\mathrm{Maps}) = \mathbb{C}\left[T^\vee \times T, \left(\frac{\alpha^\vee - 1}{\alpha^\vee - 1}\right)^{\pm 1}, \frac{\alpha^\vee - 1}{\alpha\alpha^\vee - 1}\right]^W$$
$$\|$$
$$\mathbb{C}[T \times T^\vee]$$

Spherical subalgebra of DAHA?

Perverse coherent sheaves on Gr: preserved by convolution, get (nonsymmetric) monoidal category
equivariant perverse sheaves on affine Steinberg act by Hecke correspondences on coherents on T^*Bun_G
Conj: equiv coherent sheaves on Steinbergs for G & $G^\vee$ equivalent
... under Langlands transform should be exact

$$H^{G(\mathcal{O}) \rtimes \mathbb{C}^*}(Gr_G) \overset{\text{Roma}}{\simeq} \mathcal{D}iff(G^\vee) \underset{4\times 4}{/\!/\!/\!/} N^\vee \times N^\vee$$

acts on $H^{G(\mathcal{O}) \rtimes \mathbb{C}^*}(M_{ops})$

$$\shortparallel$$

$$\mathcal{D}iff(\check{G}_{w_0}) \underset{4\times 4}{/\!/\!/\!/} N^\vee \times N^\vee \overset{?}{\simeq} \mathcal{D}iff(T^\vee)$$

$\Rightarrow$ quantum Toda Hamiltonians $H^{G(\mathcal{O}) \rtimes \mathbb{C}^*}(pt)$

$$\shortparallel$$

$$Z(U_\hbar \mathfrak{g}^\vee)$$

K-theory version $\Rightarrow$ quantum groups...

Particular solution to integrable system: char. class of zero obj.