M. Finkelberg - Uhlenbeck Spaces 10/24/07

C curve K = functions on C $\mathbb{A}$ = adeles T split torus

$F(G_{\mathcal{O}} \backslash G_{\mathbb{A}} / G_K) = F(\mathrm{Bun}_G(C))$

Induction from torus:

$$F_c(T_{\mathcal{O}} \backslash T_{\mathbb{A}} / T_K) \xrightarrow{Eis} F(G_{\mathcal{O}} \backslash G_{\mathbb{A}} / G_K)$$

Defined using

$$G_{\mathcal{O}} \backslash G_{\mathbb{A}} / G_K \xleftarrow{\pi} B_{\mathcal{O}} \backslash B_{\mathbb{A}} / B_K \xrightarrow{q} T_{\mathcal{O}} \backslash T_{\mathbb{A}} / T_K$$

$Eis = \pi_! q^*$

$$\mathrm{Bun}_G C \xleftarrow{\pi} \mathrm{Bun}_B C \xrightarrow{q} \mathrm{Bun}_T C$$

π not proper $\Rightarrow$ Drinfeld proposed a relative compactification $\overline{\pi} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$

Modified Eisenstein series, defined using $\overline{\pi}$, satisfy simple functional equation, w/o L-functions

Plücker: $G/B \hookrightarrow \prod_{\lambda \in \Lambda_G^+} \mathbb{P}(V_\lambda^*)$

Point of Bun_B $\longleftrightarrow$ $\mathcal{F}$ G-bundle + line in associated V_λ^* bundle, for all λ $\mathcal{L}_\lambda \subset \mathcal{F}_{V_\lambda^*}$

Points of $\overline{\mathrm{Bun}}_B(C)$: $\mathcal{F}$ G-bundle on C & $\mathcal{L}_\lambda \subset \mathcal{F}_{V_\lambda}$ invertible subsheaf -- i.e. allow zeros.

$G/P \hookrightarrow \prod_{\Lambda^+_{G,P}} \mathbb{P}(V_\lambda^*)$ $\lambda \in$ wall associated to P

$\Rightarrow \overline{\mathrm{Bun}}_P(C) \xrightarrow{\overline{\pi}} \mathrm{Bun}_G(C)$

If $C = \mathbb{P}^1$ $\Rightarrow$ $\mathrm{Bun}_G(C)$ contains an open part formed by trivial G-bundles. Fiber of π over this is maps $C \to G/P$

Fiber of $\overline{\pi}$ over trivial bundle is quasimaps space $\mathrm{Maps}(C, G/P) \subset \mathrm{QMaps}(C, G/P)$

Components are numbered by degree $[C] \in H_2(G/P, \mathbb{Z}) = \check{\Lambda}_{G,P}$
cocharacters of $M/[M,M]$ $\quad M = $ Levi
... only get $\check{\Lambda}^+_{G,P}$ positive coroots.

Stratification: $QMaps^\alpha(C, G/P) = \coprod_{\beta \le \alpha} Maps^\beta(C, G/P) \times C^{\alpha-\beta}$

... introduce zeros along $\check{\Lambda}^+_{G,P}$-colored divisors

$\overline{Eis} = \bar{\pi}_! q^*$ ... need to study geometry of $\widetilde{Bun_P}$ or $QMaps(C, G/P)$
$\Rightarrow$ consider "relative version", based quasi-maps

$QMaps(C, \infty; G/P, [P] \in G/P)$ $\quad$ locally base $\times$ G/P ...

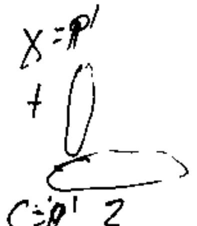
1. Factorization property: $\quad QMaps(C,\infty; G/P, P) \supset Maps^\alpha(C,\infty; G/P, P)$

$$\bar{\eta}_\alpha \searrow (C \setminus \infty)^\alpha \swarrow \eta^\alpha$$

Look at 1-dim Bruhat cells $\in B_- \backslash G/P$
$\to$ Bruhat divisors,
η^α takes a Map to the inverse image of the Bruhat divisor.
[depends on choice of B_-]
Factorization describes fibers of η:
If $D^\alpha \in (C \setminus \infty)^\alpha$ is a disjoint union $D^\beta \sqcup D^\gamma$
$$\Rightarrow (\bar{\eta}^\alpha)^{-1}(D^\alpha) = (\bar{\eta}^\beta)^{-1}(D^\beta) \times (\bar{\eta}^\gamma)^{-1}(D^\gamma)$$

2. $(\bar{\eta}^\alpha)^{-1}(\alpha \cdot c) \hookrightarrow Gr_c = G((z))/G[[z]] \quad c \in C$
$\quad \|$
intersection of semi-infinite orbits
$$\mathrm{Radical}(P^-)((z)) \cdot G \cap [P((z)) \cdot \alpha]^0 \text{ component}$$
$$\| \\ [M,M]((z)) \cdot \mathrm{Rad}\, P((z)) \cdot \alpha$$

If $\hat{\mathfrak{g}}$ is a symmetrizable K-M Lie algebra $\Rightarrow$
Kashiwara constructed (for any parabolic $\hat{\mathfrak{p}} \subset \hat{\mathfrak{g}}$)
a partial flag space $\hat{G}/\hat{P}$, scheme of infinite type



$\mathrm{Maps}^{\alpha}(C, \infty; \hat{G}/\hat{P}, \hat{P})$ is a finite-dim algebraic variety.

Take $\hat{\mathfrak{g}}$ affine Lie algebra $\mathfrak{g}((t^{-1})) \oplus \mathbb{C}K \oplus \mathbb{C}d$

$\hat{P} = G[t]$, $\hat{G}/\hat{P} = G((t^{-1}))/G[t]$ = G-bundles on X + triv at ∞

$\mathcal{G}_X \;{}^{\shortparallel}$ Grassmannian scheme

actN $\mathrm{Maps}^{a}(C, c, \mathcal{G}_X, \hat{P})$ $\overset{\bullet}{\underset{\bullet}{}}$ G-bundles on $C \times X$ + trivialization at $\infty \times X \cup C \times \infty$: for almost all $c \in C$ ($c_2 = a$) $\Big\}$ $\mathrm{Bun}^{a}_{G}(\mathbb{A}^2)$

Bundle along fibers is <u>canonically</u> trivialized

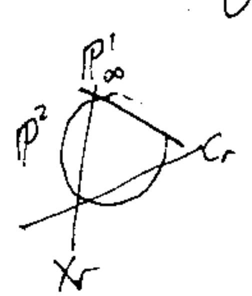
$\Rightarrow$ divisor $\eta(\mathcal{F}) \in C^{(a)}$ where $\mathcal{F}|_{c \times X}$ is nontrivial.

See along C : get formal trivializations along both divisors but don't agree.

$$Q\mathrm{Maps}^{a}(C, \infty; \mathcal{G}_X, \hat{P}) = \bigsqcup_{b \le a} \mathrm{Bun}^{b}_{G}(\mathbb{A}^2) \times (C \setminus \infty)^{(a-b)}$$

($\mathbb{P}^2$ rel $\mathbb{P}^1_\infty$ picture)

Uhlenbeck completion $U^{a} = \mathrm{Bun}^{a}_{G}(\mathbb{A}^2)$ given by Drinfeld compactification in the relative case



R = Space of pairs of transversal lines (X_r, C_r) in $\mathbb{P}^2$

$\Rightarrow$ ~~Ca~~: tautological line families $\mathcal{C} \to R$, $\mathcal{X}$

Construct relative versions $Q\mathrm{Maps}^{a}(\mathcal{C}, \infty; \mathcal{G}_{\mathcal{X}}, \hat{P}) \hookleftarrow \mathrm{Bun}_G(\mathbb{A}^2)$

$$\mathrm{Bun}_G(\mathbb{A}^2) \xrightarrow{\eta} \mathrm{Sect}(R, (\mathcal{C} \setminus \infty)^{(a)})$$

$U^{a} :=$ closure of $\mathrm{Bun}_G(\mathbb{A}^2)$ in the product

$Q\mathrm{Maps}^{a}(\mathcal{C}, \infty; \mathcal{G}_{\mathcal{X}}, \hat{P}) \times \mathrm{Sect}(R, (\mathcal{C} \setminus \infty)^{(a)})$

$$U^{a} = \bigcup_{b \le a} \mathrm{Bun}^{b}_{G}(\mathbb{A}^2) \times \mathrm{Sym}^{a-b}(\mathbb{A}^2) \qquad \text{finite type version}$$

$\mathfrak{g}^\vee$ Langlands dual affine Lie algebra: replace $\hat{P}$ by Iwahori $\hat{I}$

$$\mathrm{Maps}^\alpha(\mathbb{C}\infty;\ \hat{G}/\hat{I},\ \hat{J}) \xrightarrow{\eta^\alpha} (\mathbb{C}-\infty)^\alpha$$

$(\eta^\alpha)^{-1}(\alpha\cdot 0) = \mathcal{F}^\alpha$: Lagrangian with ∞ many components

$\coprod_\alpha \operatorname{Irr} \mathcal{F}^\alpha$ has a structure of a $\mathfrak{g}^\vee$-crystal

---- crystal $B(\infty)$ of $U^+(\hat{\mathfrak{g}}^\vee)$