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Representations of Kac-Moody algebras at critical level give candidates for geometric Langlands categories....

$\mathfrak{g}$ simple Lie algebra / $\mathbb{C}$, $\kappa$ - nonzero invariant inner product on $\mathfrak{g}$. $\Rightarrow$ central extension

$$0 \to \mathbb{C}\mathbb{1} \to \hat{\mathfrak{g}}_\kappa \to \mathfrak{g} \otimes \mathbb{C}((t)) \to 0$$

$$[A \otimes f, B \otimes g] = [A,B] \otimes fg - \left(\kappa(A,B) \operatorname{Res}_{t=0} f dg\right) \mathbb{1}$$
$$[\mathbb{1}, -] = 0$$

Consider category $\mathcal{C}_\kappa$ of $\hat{\mathfrak{g}}_\kappa$-modules $V$ s.t.

- $\mathbb{1}$ acts as Id
- $\forall v \in V \ \exists N > 0$ s.t. $\mathfrak{g} \otimes t^n \mathbb{C}[[t]] \cdot v = 0$

Critical level category ; $\kappa = \kappa_c$ critical inner product

$$\kappa_c(A,B) = -\tfrac{1}{2} \operatorname{Tr}_{\mathfrak{g}}(\operatorname{ad} A \cdot \operatorname{ad} B) = -\tfrac{1}{2} \kappa_{\text{Killing}}(A,B)$$

For $\kappa = \kappa_c$, category develops large center which we can describe, simplifying study of category.

Our category $\mathcal{C}_\kappa$ ( $U_{\kappa_c}\mathfrak{g} = U\hat{\mathfrak{g}}/(\mathbb{1}-1)$ ) is a category of modules over $U_{\kappa_c}\mathfrak{g}$, in fact for its completion

$$\tilde{U}_\kappa(\mathfrak{g}) = \varprojlim_{N > 0} U_\kappa \hat{\mathfrak{g}} / \langle \mathfrak{g} \otimes t^N \mathbb{C}[[t]] \rangle$$

1. $Z_\kappa(\hat{\mathfrak{g}}) := \operatorname{Center}(\tilde{U}_\kappa(\hat{\mathfrak{g}})) = \mathbb{C}$ for $\kappa \neq \kappa_c$

2. $Z_{\kappa_c}(\hat{\mathfrak{g}}) \cong \operatorname{Fun}(\mathrm{Op}_{{}^LG}(D^\times))$ ${}^LG$-opers on $D^\times$

This means we can view central characters $\chi : Z_{\kappa_c}(\hat{\mathfrak{g}}) \to \mathbb{C}$ as ${}^LG$-opers on $D^\times$

${}^LG$ = group of adjoint type with $\operatorname{Lie} {}^LG = {}^L\mathfrak{g}$

${}^LG$-oper on $X$ : $(\mathcal{F}, \nabla, \mathcal{F}_{{}^LB})$ $(\mathcal{F}, \nabla) = \rho$ ${}^LG$ local system

$\mathcal{F}$ ${}^LG$-bundle, $\nabla$ connection on $\mathcal{F}$,

$\mathcal{F}_{{}^LB}$ ${}^LB$ reduction of $\mathcal{F}$ (reduction to Borel) & compatibility

$$\begin{array}{ccc} \chi & \longmapsto & \rho \\ \cap & & \cap \\ {}^LG\text{-opers on } D^\times & \longrightarrow & {}^LG \text{ local systems on } D^\times \end{array}$$

On punctured disc any $\rho$ lifts to an oper $\chi$
So to local system $\rho$ can assign category
$\mathcal{C}_{\kappa_c,\chi}$ - subcategory of $\mathcal{C}_{\kappa_c}$ whose objects are $\hat{\mathfrak{g}}_\kappa$-modules with central character given by $\chi$ ..
To what extent does this depend on choice of $\chi$ over fixed $\rho$?

Cannot currently describe full $\mathcal{C}_{\kappa_c,\chi}$ but only can conjecture shape of Iwahori-integrable parts.

Why assign category to local system?
Langlands: $F$ local nonarchimedean field,
$\rho: \mathrm{Gal}\, \bar{F}/F \longrightarrow {}^LG \rightsquigarrow$ a rep of $G(F)$
(or an L-packet thereof...)
Geometric version:
${}^LG$-local system on $D^\times \rightsquigarrow$ a rep of $G((t))$ ?
well not so many representations of ~~this~~ group...
first should replace by K-M group,
but not many reps: integrable reps form a semisimple category but local systems are very complicated.

So replace $G((t))$ by $\mathfrak{g}((t))$ or $\hat{\mathfrak{g}}$ representations

If $(V, \alpha)$ is a rep of $\hat{\mathfrak{g}}_\kappa$ $\quad \alpha: \hat{\mathfrak{g}}_\kappa \to \mathrm{End}\, V$
and $g \in G((t))$ can construct new representation $(V, Ad_g^\times(\alpha))$

- if $(V,\alpha)$ integrable would get isomorphic rep, but usually get different reps.
So get action of $G((t))$ on Category of $\hat{\mathfrak{g}}$-modules!
Moreover infinitesimally trivial action: Lie algebra of $G((t))$ preserves objects (at least projectively)

— Want to replace reps of $G(F)$ by certain actions of $G((t))$ on categories ...

| Classical story $G(F)$ | | Kac-Moody $\hat{\mathfrak{g}}$ |
| --- | --- | --- |
| Representation | $M$ | Category $\mathcal{C}$ of $\hat{\mathfrak{g}}$-modules |
| K compact subgroup (spherical vectors for max cpt) compact invariants | $M^K$ | Category of Harish-Chandra $(\hat{\mathfrak{g}}, K)$-modules |
| Hecke algebra acts on invariants | $\mathcal{H}(G(F), K)$ | Categorical Hecke algebra: $K$-equivariant $\mathcal{D}$-modules on $G((t))/K$ |

Does this RHS make sense for $K$ any compact open subgroup?

Drinfeld

Want "monoidal dg category" of $K$-equivariant $\mathcal{D}$-modules on $\widehat{G}/K$.

-- need to work with nonholonomic $\mathcal{D}$-modules - can speak about families etc -- so only one pushforward $p_*$ & one pull back $p^!$, so not much choice about "which convolution" - holonomic have $*$ & $!$ convolutions

Wt To identify these with a category of $\mathcal{O}$-modules with $\otimes$ ...

Model situation: $\mathcal{D}$-mod on $\mathbb{A}^1$ with convolution

Fourier transform identifies this with $\mathcal{D}$-mod on $(\mathbb{A}^1)^\vee$

Two convolutions $\longleftrightarrow$ two tensor products on these $\mathcal{D}$-modules - usual & ! tensor products.

So two convolutions more generally might be expected to correspond to 2 convolutions on $\mathcal{O}$-mod

Fin dim problem: define convolution on category of coherent $\mathcal{D}$-mods on alg. group $G$?

no: proper pushforward OK for coherent $\mathcal{D}$-mods but general pushforward preserves holonomicity but

not coherence (for holonomic need to use b-functions)
Once we leave holonomics coherence not preserved — so either work with noncoherent or think of convolution as functor $\mathrm{Coh}\,\mathcal{D}\text{-mod} \times \mathrm{Coh}\,\mathcal{D}\text{-mod}$

$$\mathrm{Coh}\,\mathcal{D}\text{-mod} \times \mathrm{Coh}\,\mathcal{D}\text{-mod} \to \mathrm{Ind}(\mathrm{Coh}\,\mathcal{D}\text{-mod}) = \mathcal{D}\text{-mod}$$

($\mathcal{D}$-mod = ind coherent D-mod)
primary objects

Direct image of coherent $\mathcal{D}$-module is a complex
so should work with dg modules.
Ind object is dg functor $\mathcal{D}$-mod $\longrightarrow$ complexes
so bifunctor $\mathrm{Coh} \times \mathrm{Coh} \longrightarrow \mathrm{Ind\,Coh}$ is actually a trifunctor given by $\mathcal{D}$-module on

$$G \times G \times G \supset \underbrace{\{ g_1 g_2 = g_3 \}}_{Y}$$

Consider $\mathcal{O}_Y$ as $\mathcal{D}_{G\times G\times G}$-module $\Rightarrow$ kernel of convolution map for $\mathcal{D}$-mod on $G$ • • •

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Triangulated category $\mathcal{D}$-mod$(\mathcal{G}r)$ $\supset$ subcategory of compact objects : coherent $\mathcal{D}$-module, ~~supported~~
~ get well defined trifunctor as above
- can start to think about such questions of convolution.

[ $l$-adic sheaves: don't have true moduli of local systems but do have formal neighborhoods ... formally don't need nonholonomic $\mathcal{D}$-modules — our ~~Hecke~~ $(G, \mathcal{D})$-modules on formal nbhds are holonomic ]

Nonholonomic setting: can just work with single $\mathcal{D}$-module $\mathcal{D}_G$ itself, close to representation theory
— think seriously about finite dimensions!

Hecke functors have nontrivial Exts (Ginzburg) ... so more nonclassical structure

The Exercise

Problem: 1. dg category of spherical $\mathcal{D}$-modules on $G(K)/G(O)$ as monoidal dg category (whatever that means) is symmetric on abelian category level --- what about dg level? can define some braiding but not clear in what sense it is symmetric? --- need 2-operad for such dg categories

2. How to describe the derived Satake category concretely in this language, in terms of ${}^L G$? --- homotopical problem

What are these Exts of spherical sheaves for? Given Hecke eigensheaf $\Longrightarrow$ universal deformation

$$\mathcal{D}\text{-mod on } Bun_G = \int_{LocSys} \text{automorphic } \mathcal{D}\text{-modules}$$

Hecke Categories corresponding to different local systems are more or less canonically isomorphic

Working with dg categories / Exts should be able to identify these categories for infinitesimally nearby local system

So dg structure is for deformation theory. (?)

$Ext^2$'s responsible for everything ...

Conjecturally $pt \times_G pt$ ~~$\mathfrak{g}$~~ as dg stack
$pt \to G$ by unit element

Bezrukavnikov Ginzburg: 1. $H^*_{G(O)}(Gr, IC_\lambda)$ described as a module over $H^*_{G(O)}(pt)$
$= H^*_G(pt) = \mathcal{O}(\mathfrak{h}/W)$
$= \mathcal{O}(\mathfrak{h}^{\vee *}/W)$

2. proved $Ext_{G(O)}(IC_\lambda, IC_\mu)$

$$\xrightarrow{\sim} Hom_{H^*_{G(O)}(pt)}(H^*_{G(O)}(IC_\lambda), H^*_{G(O)}(IC_\mu))$$

Corollary $Ext^*_{G(O)}(IC_\lambda, IC_\mu) = Hom_{Coh^{G^\vee}(\mathfrak{g}^{\vee *})}(V_\lambda \otimes \mathcal{O}, V_\mu \otimes \mathcal{O})$

equivariant coherent ←---- sheaves

Hom between equivariant vector bundles on dual vector space to $G^\vee$. RHS graded by weights of $\mathbb{G}_m$ action on vector space $(\mathfrak{g}^\vee)^*$
LHS graded by cohomological degree

$\mathrm{Ext}^{2i} \longleftrightarrow$ degree $i$ hom.

Reformulation: $\mathrm{Ext}^{2\cdot}_{G(\mathcal{O})}(IC_0, \bigoplus_\lambda IC_\lambda \otimes V_\lambda^*) \circlearrowleft G^\vee$
$\qquad\qquad\qquad //$ $\qquad$ $\delta$-functions $\qquad$ regular rep

$\mathrm{Sym}^\bullet(\mathfrak{g}^\vee) \circlearrowleft G^\vee$

~~Drinfeld conjectures~~: Formality property: work with Weil sheaves, have purity of Frobenius eigenvalues ($\mathrm{Ext}^i$ has wt $i$) $\Longrightarrow$ isomorphism of Ext extends to equivalence of categories (triangulated)

$D_{G(\mathcal{O})}(\mathrm{Gr}) =$ dg-modules over dg algebra with 0 differential: $G^\vee \ltimes \mathrm{Sym}(\mathfrak{g}^\vee[-2])$

... we want something like $G^\vee$-equiv sheaves on $(\mathfrak{g}^\vee)^*$ but cohomological grading should correspond to $\mathbb{G}_m$ action.

Koszul duality: $\text{dg-mod}(\mathrm{Sym}\ \mathfrak{g}^\vee[-2]) \simeq \text{dg-mod}(\Lambda^\bullet \mathfrak{g}^{\vee *}[1])$

but $\Lambda^i \mathfrak{g}^{\vee *} = \mathrm{Tor}^i_{\mathcal{O}(\mathfrak{g}^\vee)}(k, k)$ $\qquad$ $k$ supported at 0

so RHS = coherent sheaves on dg scheme

$\{0\} \overset{L}{\underset{\mathfrak{g}^\vee}{\times}} \{0\}$ $\qquad$ easiest example of dg scheme

Theorem
Conjecture
(Drinfeld).. $\quad D_{G(\mathcal{O})}(\mathrm{Gr}) \xrightarrow{\sim} D\mathrm{Coh}_{G^\vee}\left(\{0\} \overset{L}{\underset{\mathfrak{g}^\vee}{\times}} \{0\}\right)$

as monoidal (D$^b$) categories $\qquad$ $\overset{\shortparallel}{DG\text{-mod}^{G^\vee}}\left(\mathcal{O}_{\{0\} \overset{L}{\times}_{\mathfrak{g}^\vee} \{0\}}\right)$

[ For $X \to Y$ proper, $\mathcal{D}Coh(X \overset{L}{\underset{Y}{\times}} X)$
is monoidal category under convolution ]

<u>Remarks</u> 1. This fits in a series of conjectures identifying different categories of sheaves with categories of coherent sheaves on (dg) schemes associated with Langlands dual groups
- describe $D_I(Gr)$, $D_I(Fl)$

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<u>Frenkel, continued</u>

Study categories of $(\hat{\mathfrak{g}}_{\kappa_c}, K)$-modules with some fixed central character.
$\chi : Z(\hat{\mathfrak{g}}_{\kappa_c}) \to \mathbb{C}$ $\quad \chi \in Op_{{}^LG}(D^\times)$

<u>Example</u> $K = G[[t]]$ max. compact, $\chi \in Op_{{}^LG}(D) \subset Op_{{}^LG}(D^\times)$

<u>Claim:</u> 1. The category $(\hat{\mathfrak{g}}_{\kappa_c}, G[[t]])\text{-mod}_\chi$ is $\overline{\phantom{x}}$ : $\mathcal{C}_{G[[t]],\chi}$ equivalent to the category of vector spaces.

The irreducible $(\hat{\mathfrak{g}}_{\kappa_c}, G[[t]])_\chi$ module is constructed as follows:
Consider vacuum module $V = \mathrm{Ind}^{\hat{\mathfrak{g}}_{\kappa_c}}_{\mathfrak{g}[[t]] \oplus \mathbb{1}} \mathbb{C}$,
($\mathfrak{g}[[t]]$ acts by 0, $\mathbb{1}$ by 1)

$$\begin{array}{ccc} Z(\hat{\mathfrak{g}}_{\kappa_c}) & \longrightarrow & \mathrm{End}_{\hat{\mathfrak{g}}_{\kappa_c}} V \\ \| & & \text{is} \\ \mathrm{Fun}(Op_{{}^LG}(D^\times)) & \twoheadrightarrow & \mathrm{Fun}(Op_{{}^LG}(D)) \end{array}$$

So take quotient $V_\chi = V / \mathrm{Im}\, I_\chi$
$I_\chi$ - max ideal of $\chi$ in $\mathrm{End}_{\hat{\mathfrak{g}}_{\kappa_c}} V$

This module is irreducible $\forall \chi \in Op_{{}^LG}(D)$ & each object in the category $\mathcal{C}_{G[[t]],\chi}$ is a direct sum of copies of $V_\chi$.

Now allow $\chi$ to vary within $Op_{{}^LG}(D)$ :

Study category $\mathcal{C}_{G[[t]], reg}$ : $(\hat{\mathfrak{g}}_{\kappa_c}, G[[t]])$-modules

s.t. $Z_{\kappa_c}(\hat{\mathfrak{g}})$ acts through its quotient $\mathrm{Fun}\, Op_{^LG}(D)$

Claim 1'. $\mathcal{C}_{G[[t]], reg} \simeq \mathrm{Fun}\, Op_{^LG}(D)$ - modules

$$\begin{aligned} M &\longmapsto \mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(\underset{Z(\hat{\mathfrak{g}})}{\underset{\circlearrowright}{V}}, M) \\ V \underset{\mathrm{Fun}(Op\,D)}{\otimes} \mathcal{F} &\longleftarrow \mathcal{F} \end{aligned}$$

these two functors are exact, & set up equivalence

$V_\chi \longleftrightarrow$ skyscraper at $\chi$

Describe support in Spec $\underset{Op\,D^\times}{Z}$ of $G[[t]]$-equivariant sheaves

... it is a disjoint union of subschemes one of which is regular opers --- others labeled by dominant weights of Langlands dual group --- so condition of $G[[t]]$-equivariance & regularity of oper central character are not independent condition, first almost implies second ... components all with same

We are losing extension data of our modules: should look at $\chi$-isotypic components, supported over $\chi$ set theoretically but not nec. scheme theoretically: information in formal neighborhood of $Op\,D \subset Op\,D^\times$ ...

Example 2 $K = I$, Iwahori subgroup

$\mathcal{C}_{I, nOp_{^LG}}$ : objects are $(\hat{\mathfrak{g}}_{\kappa_c}, I)$-modules s.t. $Z_{\kappa_c}(\hat{\mathfrak{g}})$ acts through functions on

nilpotent opers $nOp_{^LG} \subset Op_{^LG}(D^\times)$ )

... set theoretic support in $Op(D^\times)$ of $(\hat{\mathfrak{g}}, I)$-modules is a union of components, one of which is $n\,Op_{{}^LG}$ $\boxed{\text{(all components} \Leftrightarrow \text{unipotent monodromy}}$ )

$n\,Op_{{}^LG}$ : opers with unipotent monodromy with additional condition: extend to entire disc (Deligne extension) $\Rightarrow$ connection with regular singularity.

~~Require residue~~ ~~a principal nilpotent of~~ $\cancel{{}^L\mathfrak{g}}$,

Restriction lies in ${}^L\mathfrak{n}$, require oper condition at 0 also

$\mathfrak{g} = sl_2$    $PGL_2$ opers = projective connections

$\partial_t^2 - v(t) : \Omega^{-1/2} \to \Omega^{3/2}$    $v(t) = \sum v_n t^n$ Laurent

$Op_{PGL_2}(D)$ : $v(t) = \sum_{n \geq 0} v_n t^n$ Taylor series

$n\,Op_{PGL_2}$ : $v(t) = \sum_{n \geq -1} v_n t^n$ mild singularities

$n\,Op_{{}^LG}(D)$ : $\partial_t + \begin{pmatrix} a & b \\ \underline{1} & -a \end{pmatrix} / t$ doesn't vanish at 0 either

as opposed to $\partial_t + \begin{pmatrix} a & b \\ t^n & -a \end{pmatrix}$ where $\begin{pmatrix} & \\ \bullet & \end{pmatrix}$ entry vanishes of zero

Remark   Let $M = \mathrm{Ind}_{\hat{\mathfrak{b}} \oplus \mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_{\kappa_c}} \mathbb{C}_1$   Verma module

where $\hat{\mathfrak{b}} = \mathrm{Lie}\, I$ · Iwahori

$$\mathrm{End}_{\hat{\mathfrak{g}}_{\kappa_c}} M = \mathrm{Fun}(n\,Op_{{}^LG})$$

$M$ is free over this module of endomorphism.

So can define functors like before between representations & sheaves on $n\,Op_{{}^LG}$.

However for fixed $\chi \in n\,Op_{{}^LG}$ category of reps with central character $\chi$ is more complicated than Vect which we had before

– Verma modules are not irreducible!

$\rightarrow$ get natural subbundles for $M_\chi$
(Note $M = \mathrm{Ind}_{\mathfrak{g}(\mathcal{O}) \oplus \mathbb{1}}^{\hat{\mathfrak{g}}} \underline{M}$ $\rightsquigarrow$ fin dim ~~$\Phi$~~ $\mathfrak{g}$'s Verma mod

So want to introduce a variety so that quasicoherent sheaves on it are $\simeq$ modules with fixed central character

Seek $Y$ scheme s.t. $D^b(\mathcal{O}_Y\text{-mod}) \simeq D^b(\mathcal{C}_{I, n\mathrm{Op}_{^L G}})$

$$\begin{array}{c} Y \\ \downarrow \\ n\mathrm{Op}_{^L G} \end{array}$$

<u>Clue</u>: Construction of the "principal series" representations in $\mathcal{C}_{I, n\mathrm{Op}_{^L G}}$ — the Wakimoto modules.

$\underline{\mathfrak{g} = \mathfrak{sl}_2}$ $\quad \{\partial_t - u(t)\} = \mathrm{Conn}_{\Omega^{\frac{1}{2}}}(D^\times)$ $\quad u$

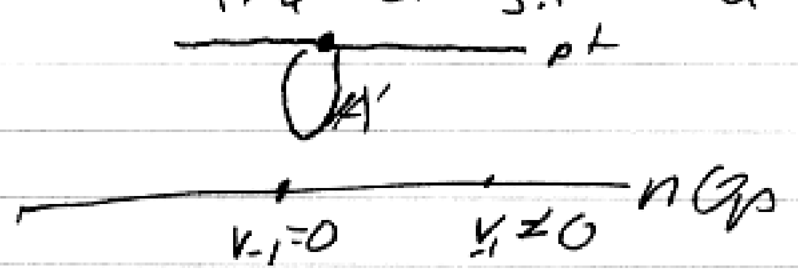
Miura $\{(\partial - u)(\partial + u)\}$ $\quad \{\partial_t^2 - v(t)\} = \mathrm{Op}_{PGL_2}(D^\times)$ $\quad u^2 - u'$

Any module $\mathcal{F}$ over $\mathrm{Fun}(\mathrm{Conn}_{\Omega^{\frac{1}{2}}}(D^\times))$ gives rise to an $\hat{\mathfrak{g}}_\kappa$-module $W(\mathcal{F})$

$\mu \in \mathrm{Conn}_{\Omega^{\frac{1}{2}}}(D^\times)$, skyscraper $\delta_\mu \longmapsto$ Wakimoto module $W_\mu$

So get family of modules parametrized by $\mathrm{Conn}_{\Omega^{\frac{1}{2}}}(D^\times)$

Look for these connections lying over nilpotent opers
... find $u$ s.t. $u^2 - u'$ has pole of order $\leq 1$.




: $\mathbb{A}^1 \times$ regular opers
$\cup$ $n\mathrm{Op}$ with $v_{-1} \neq 0$

Parameters of Wakimoto modules $\longleftrightarrow$ points of our scheme $Y$, & skyscrapers $\leftrightarrow$ Wakimotos

Actually situation is doubled: $\mathbb{P}^1$ over regular opers + double line over nilpotent opers

$$\begin{array}{ccc} nMOp_{{}^LG} & \longrightarrow & {}^L\tilde{n}/Ad\,{}^LB \supset Sp_x = \{B \ni x\} \\ \downarrow & & \downarrow \quad\quad \downarrow \\ nOp_{{}^LG} & \xrightarrow[res]{} & {}^Ln/Ad\,{}^LB \ni x \end{array}$$

Grothendieck-Springer resolution

nilpotent Miura Opers

Springer fiber

Actually parameters of Wakimoto modules are disjoint union of two components $\underline{\quad} \sqcup \underline{\quad}$

but have nontrivial extensions gluing these two together in rep theory --- so have to glue these two together geometrically $\Rightarrow$ nilpotent Miura opers

$nMOp_{{}^LG} = (\underbrace{\mathcal{F}, \nabla, \mathcal{F}_{{}^LB}}_{oper}, \underbrace{\mathcal{F}'_{{}^LB}}_{\text{horizontal flag}})$

--- replace $Conn_{\Omega^\rho}$ by $MOp$, glues together the different components of Conn (Cartan connections)

--- Wakimoto parametrized initially by Cartan connections, but Miura opers glue these together.

Now have Springer fibers appears..

<u>Conjecture</u> There is an equivalence of categories

$$D^b(\mathcal{O}_{nMOp_{{}^LG}}\text{-mod}) \xleftrightarrow{\sim} D^b(\mathcal{C}_{I\text{-monodromic}}, nOp_{{}^LG})$$

skyscraper at $\mu \in nMOp_{{}^LG}$ $\longleftrightarrow$ Wakimoto $W_\mu$ $(\mu \in Conn^{{}^LT})$

<u>Requirements</u>! Both live over $nMOp_{{}^LG}$ & carry actions of categorical Hecke algebra

$D^b_I(Fl)$ : $\underline{D^b(\mathcal{C}_{I\text{-mon}})} \simeq D^b(Coh_{{}^LG}(\text{Steinberg}))$

Bezrukavnikov so acts on $\underline{\mathcal{O}_{nMOp}\text{-mod}}$!

will really need equivariant derived category

[--- Spherical case $\mathcal{O}_{Op_{{}^LG}(D)}\text{-mod} \longleftrightarrow \mathcal{C}_{G[[t]],res}$

$$Rep\,{}^LG \xrightarrow[\text{Spherical Hecke}]{\sim} Perv_{G(\mathcal{O})}(Gr)$$ ]

Local $\to$ Global : $\mathcal{C}_{I,\, n\mathrm{Op}_G}$ localizes on $\mathrm{Bun}_G$ :

$(\hat{\mathfrak{g}}_{x_c}, I)$-mod $\rightsquigarrow_{\Delta}$ $\mathcal{D}$-module on $\mathrm{Bun}_{G,I,x}$

B-B localization — parabolic structure at $x$

$\chi \in n\mathrm{Op}_{^LG}(D)$ :

$M \in \mathcal{C}_{I,\chi \in n\mathrm{Op}} \longrightarrow \Delta(M)$

Conjectural Claim: $\Delta(M) = 0$ unless $\chi$ extends to a regular oper on $X - x$

2. $\Delta(M) \neq 0$ is a Hecke eigensheaf with respect to the oper extending $\chi$.

Miura ~~opers~~ opers $\rightsquigarrow$ chiral differential operators on $G/B$:

Wakimoto modules $\leftrightarrow$ modules over vertex algebra of chiral diffops on big cell :

Theorem: twistings on the big cell are Miura opers ... space is space of chiral differential operators

Cor. Module over chiral diffops corresponding to open subset of flag variety carries canonically Kac-Moody module structure : just need to specify twistings

Two constructions of objects: Wakimotos & inductions / central characters. First is $\infty/2$ flag picture, latter lives on affine flags, & somehow they are identified !

Verma module with h.w. $-2\rho$ ($\leftrightarrow w_0$) -- point in usual flags ---

$\Rightarrow$ induced module's ~~central~~ character will be Wakimoto module corresponding to central component $0$

Fin dim situation! have versions of Wakimoto using Schubert correspondences & different twistings $\longleftrightarrow$ apply intertwining functors to usual Vermas.

Is there a version here relating affine flags & $\infty/2$ flags?

This conjecture says Bezrukavnikov's category $\mathcal{C}_p$ agrees with guess coming from critical level modules over Kac-Moody
-- coherent sheaves on Springer fibers $\leftrightarrow$ ~~coded~~ critical modules...

What's relation of sheaves on $\infty/2$ flags & reps?
Wakimoto are very simple examples of this...

Is there a global sections functor to $\mathfrak{g}_{x_c}$-modules from "$\mathcal{D}$-modules on $\infty/2$ flags"?

Gluing of components of $\mathrm{MOp}$ $\leftarrow\rightarrow$ gluing of $\mathcal{D}$-modules corresponding to Wakimoto living on different Schubert cells

[$\infty/2$ flags only have to do with case of no monodromy]