E. Frenkel - Geometric Langlands   Davis 4/25/98

$$\left\{ \begin{array}{c} n\text{-dim} \\ \text{reps of } Gal(\bar{F}/F) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{automorphic} \\ \text{reps of } GL_n(\mathbb{A}) \end{array} \right\}$$

F global field e.g. $\mathbb{F}_q(X)$, or number field

$\mathbb{A}$ adeles $= \prod' F_v$ restricted product of completions

Examples $n=1$ Abelian Class Field Theory

1-dim reps of $Gal(\bar{F}/F)$ - factors through $Gal(\bar{F}/F)_{ab}$

$Gal(\bar{F}/F)_{ab} \simeq GL_1(F) \backslash GL_1(\mathbb{A})$

e.g. $F = \mathbb{Q}$ Kronecker - Weber

$Gal(\bar{\mathbb{Q}}/\mathbb{Q})_{ab} \simeq Gal(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \simeq \prod_p GL_1(\mathbb{Z}_p)$

$n=2$ special case: Taniyama-Shimura - 2d reps of Galois group associated to elliptic curves.

Let $F = \mathbb{F}_q(X)$ rational fns on X smooth proj $/\mathbb{F}_q$.

$x \in X \rightsquigarrow$ completion of F $F_x \simeq \mathbb{F}_{q_x}((t_x))$, $q_x = q^{\deg x}$

$\mathcal{O}_x \simeq \mathbb{F}_{q_x}[[t_x]]$ compact

G reductive group split $/F_x$ (e.g. GL_n)

$G(\mathcal{O}_x) \subset G(F_x)$. $G(\mathbb{A}) = \prod'_{x \in X} G(F_x)$

$= \{ (g_x) \mid g_x \in G(\mathcal{O}_x)$ all but fin many $x \}$

Fix $x \in X$. π_x rep of $G(F_x)$ in a $\bar{\mathbb{Q}}_\ell$-vector space.

1. Smooth: stabilizer of any vector open in $G(F_x)$
2. Unramified: $\exists v_x \in \pi_x$, $G(\mathcal{O}_x) \cdot v_x = v_x$

Theorem (Satake, Langlands)

$$\left\{ \begin{array}{c} \text{irred smooth} \\ \text{unramified reps of } G(F_x) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{semi-simple conj} \\ \text{classes in } {}^L G(\bar{\mathbb{Q}}_\ell) \end{array} \right\}$$

field of def of representation

G split reductive $/F$

T - maximal torus $\simeq (F^\times)^n$

Root data: P lattice of characters

$P^\vee$ 1-param subgps $F^\times \to T$

$\Delta \subset P$, $\Delta^\vee \subset P^\vee$ roots & coroots

$$\overset{G}{(P, P^\vee, \Delta, \Delta^\vee)} \longleftrightarrow \overset{{}^L G}{(P^\vee, P, \Delta^\vee, \Delta)}$$

GL_n	GL_n
SL_n	PGL_n
SO_{2n+1}	Sp_{2n}

$F \subset \mathbb{A} = \prod'_{x\in X} F_x$. L - loc. const functions on $\cancel{G(F)\backslash G(\mathbb{A})}$

$L^\circ \subset L$ cuspidal functions (integrals over N-orbits vanish).
$G(\mathbb{A})$ acts on L°.
A rep π of $G(\mathbb{A})$ is called <u>automorphic</u> if it occurs in L°.
Unramified reps $\pi = \otimes' \pi_x$ $x \in X$, each $\pi_x \ni v_x$.

Question: which $\otimes'_{x\in X} \pi_x$ are automorphic?

Recall $\pi_x \longleftrightarrow \gamma_x$ s.s. conj class in ${}^LG$.
which condition should $\{\gamma_x\}_{x\in X}$ satisfy for automorphy?
<u>Langlands</u>: $\{\gamma_x\}_{x\in X}$ should come from $\sigma: Gal(\bar{F}/F) \to {}^LG$.

$F_Y = \mathbb{F}_q(Y)$ - alg. extension of $F = \mathbb{F}_q(X)$. $Gal(F_Y/F)$ acts by deck transformations. $Gal(\bar{F}/F)_{un}$ - max unramified quotient of $Gal(\bar{F}/F)$
- analog of $\pi_1(X)$.
Each $x \in X \Longrightarrow$ Frobenius Fr_x conj class in $Gal(\bar{F}/F)_{un}$; $\alpha \to \alpha^q$ on residue field
$\Longrightarrow$ get collections of conj. classes from Fr_x

<u>Langlands Conjecture</u> $G(\mathbb{A})$-rep $\otimes' \pi_x$ is auto iff
$\exists\ \sigma: Gal(\bar{F}/F)_{un} \longrightarrow {}^LG$ s.t. $\gamma_x = \sigma(Fr_x)$ $\forall x \in X$.

<u>Note</u> 1. If $G \neq GL_n$, σ not uniquely def. by $\sigma(Fr_x)$. (Chebotarev for GL_n). Also mult$(\otimes' \pi_x)$ may be >1.
2. If $\otimes' \pi_x$ is automorphic
$\Longrightarrow \otimes v_x$ gives <u>automorphic function</u> on $G(F)\backslash G(\mathbb{A})/G(\mathcal{O})$
- eigenfunction of Hecke operators.

<u>Geometric Langlands</u> X complex curve.
$Gal(\bar{F}/F)_{un} \longrightarrow \pi_1(X)$.

$G(F)\backslash G(\mathbb{A})/G(\mathcal{O}) \longrightarrow Bun_G(X)$ moduli of G-bdles on X

$\left\{ \begin{array}{c} \text{homomorphisms} \\ \pi_1(X) \longrightarrow {}^LG \end{array} \right\} \overset{?}{\longleftrightarrow} \left\{ \begin{array}{c} \text{holonomic systems} \\ \text{of diff eqn on } Bun_G(X) \end{array} \right\}$

D-mods

(Automorphic function $\longleftrightarrow$ solutions of the equations..)

<u>Local picture</u> Reps of $\mathfrak{g}(\mathbb{C}((t)))$
& its central extension $\hat{\mathfrak{g}}$

$$0 \to \mathbb{C}K \to \hat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 0$$

Analog of unramified rep: $\hat{\mathfrak{g}}$-rep in V, s.t.
1. K acts by scalar $k \in \mathbb{C}$ 2. $\exists v \in V$, $\mathfrak{g}[[t]] \cdot v = 0$

If $k \neq -\frac{1}{2} \Rightarrow \exists!$ such..

<u>Theorem</u> (F-F)

$$\left\{\begin{matrix}\text{irred unramified} \\ \text{reps of } \hat{\mathfrak{g}},\ k = -\frac{1}{2}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}{}^L\mathfrak{g}\text{-opers} \\ \text{on formal disc}\end{matrix}\right\}$$

<u>Opers</u> sl_2 oper on a curve $X/\mathbb{C}$: rank 2 bdle $\mathcal{F}$,
line $\mathcal{F}_1 \subset \mathcal{F}$, $\nabla: \mathcal{F} \to \mathcal{F} \otimes \omega_X$

$$\mathcal{F}_1 \to \mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes \omega_X \to (\mathcal{F}/\mathcal{F}_1) \otimes \omega_X$$

Locally $\nabla = \partial_t + \begin{pmatrix} & q(t) \\ 1 & \end{pmatrix}$

$\Leftrightarrow$ proj. connection $\partial_t^2 - q(t): \omega^{-\frac{1}{2}} \to \omega^{\frac{3}{2}}$

Change coord. : $t = \phi(s)$

$\Rightarrow \partial_s^2 - q(\phi(s))(\phi'(s))^2 + \frac{1}{2}\{\phi, s\}$ ← Schwarzian derivative

. . . . Sl_n oper . . . . .

$$\partial_t^n - q_1 \partial_t^{n-2} - \dots : \omega^{-\frac{n-1}{2}} \to \omega^{\frac{n+1}{2}}$$

${}^LG$ oper $\Rightarrow$ ${}^LG$ bdle w/connection $\Rightarrow$ hom. $\pi_1(X) \to {}^LG$.

<u>Global Picture</u> X complex curve Suppose for each
$x \in X \to {}^L\mathfrak{g}$ oper ρ_x on D_x
$\Rightarrow$ V_{ρ_x} irred unramified rep.

Call $\bigotimes'_{x \in X} V_{\rho_x}$ automorphic if $\exists\, \mathfrak{g}(\mathbb{C}(X))$-invariant functional on $\bigotimes V_{\rho_x}$ (instead of $G(F)$ invariant before)

Thm (B-D) $\bigotimes V_{\rho_x}$ is automorphic iff $\exists$ global oper ρ s.t. $\rho_x = \rho|_{D_x}$.

$\Rightarrow \sigma: \pi_1(X) \to {}^LG$. Construct system of diff eqs on $\mathrm{Bun}_G \Rightarrow \Delta_\sigma$. Check that this is Hecke eigensheaf.