E. Frenkel - Geometric Langlands & Affine Kac-Moody Algebras 4/17/02

Local Langlands - reps of groups like $GL_n(\mathbb{Q}_p)$, $GL_n(\mathbb{F}_q((t)))$
$\rightsquigarrow$ reps of $0 \to \mathbb{C}K \to \hat{\mathfrak{g}} \to \mathfrak{g} \otimes \mathbb{C}((t)) \to 0$
as complex analogue.

Local Langlands correspondence relates such representations ($GL_n(K)$)
to n-dimensional representations of $Gal(\bar{K}/K)$
(or Weil group). Try to do something similar for representations of $\hat{\mathfrak{g}}$: Galois datum $\rightsquigarrow$ bundles with connection on punctured disc etc.

Outline of Langlands picture, global setting:
F = global field - i.e. [number field or] function field $F = \mathbb{F}_q(X)$
X smooth projective curve $/\mathbb{F}_q$

$\bar{F}$: separable closure of F

$$\left\{\begin{array}{l}\text{irred } n\text{-dim reps}\\ \text{of Weil group}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{cuspidal automorphic}\\ \text{reps of } GL_n(\mathbb{A})\end{array}\right\} \quad [\text{Lafforgue}]$$

$x \in X \Rightarrow$ completion K_x of F $\quad K_x \simeq \mathbb{F}_{q_x}((t)) \supset \mathbb{F}_{q_x}[[t]] \simeq \mathcal{O}$
$\mathbb{A} = \prod'_{x \in X} K_x$ almost everywhere in $\mathcal{O}_x \subset K_x$.

Automorphic : can be realized in functions on $GL_n(F) \backslash GL_n(\mathbb{A})$
If π is such & $v \in \pi$ s.t. $K \cdot v = v$ $\Rightarrow$
function on $GL_n(F) \backslash GL_n(\mathbb{A}) / K$

Replace GL_n by G - split reductive group $/\mathbb{F}_q$ $\Rightarrow$
reps of $G(\mathbb{A})$ $\longleftrightarrow$ $Gal(\bar{F}/F) \longrightarrow {}^LG$
${}^LG$ Langlands dual group : $G \supset T$ $\quad X^*(T)$ characters $\quad X_*(T)$ cocharacters
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup \qquad\qquad\qquad \cup$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Delta$ roots $\qquad$ coroots $\Delta^\vee$

- switch root data $\Rightarrow$ ${}^LG$.
simply connected $\leftrightarrow$ adjoint, $B \leftrightarrow C$ etc.
Much more complicated for $G \neq GL_n$ ...
$G(\mathbb{A}) = \prod'_{x \in X} G(K_x)$ $\qquad Gal(\bar{F}/F) \hookleftarrow Gal(\bar{K}_x/K_x)$

X curve $/\mathbb{C}$: Galois group $\rightsquigarrow$ fundamental group :
$\pi_1(X) \longrightarrow {}^LG$ $\Rightarrow$ ${}^LG$-local systems $\leftrightarrow$ ${}^LG$-bundle + (flat) connection

Unramified case $\rightsquigarrow$ functions on $G(F)\backslash G(\mathbb{A})/G(\mathcal{O})$

$K = G(\mathcal{O}) = \prod_x G(\mathcal{O}_x)$ max compact which are eigenfunctions for Hecke operators

functions / $\mathbb{F}_q$ $\rightsquigarrow$ constructible sheaves $\xrightarrow{\text{Riemann-Hilbert}}$ $\mathcal{D}$-modules

Weil $\quad G(F)\backslash G(\mathbb{A})/G(\mathcal{O}) = \{$ iso classes of principal G-bundles on X $\}$

... should be set of points of moduli space of G-bundles $\rightsquigarrow$ really algebraic stacks: Bun_G

$\rightsquigarrow$ conjecture (Drinfeld, Laumon, Beilinson)

${}^LG$-bundles on X with connection $\longleftrightarrow$ $\mathcal{D}$-modules on Bun_G with Hecke condition

<u>Localization functor</u>: Rep of Vir $\longrightarrow$ $\mathcal{D}$-module on moduli of curves, fibers are spaces of conformal ...

Rep of $\hat{\mathfrak{g}}$ $\longrightarrow$ $\mathcal{D}$-module on Bun_G

Integrable vacuum rep of level k L_k $\Longrightarrow$ sheaf of vacua of WZW model, with connection – projectively flat (<u>twisted</u> $\mathcal{D}$-modules)

To construct Hecke eigensheaves need reps of $\hat{\mathfrak{g}}$ of critical level where Sugawara breaks down:
$\rightsquigarrow$ study reps of $\hat{\mathfrak{g}}$ at critical level

<u>Thm</u> (Feigin-Frenkel) Let $\tilde{U}_{-h^\vee}(\hat{\mathfrak{g}}) = \varprojlim_{N>0} U_{-h^\vee}(\hat{\mathfrak{g}}) / U_{-h^\vee}(\hat{\mathfrak{g}})\, \mathfrak{g}\otimes t^N\mathbb{C}[[t]]$

[acts on "smooth" representations: all vectors killed by $\mathfrak{g}\otimes t^N\mathbb{C}[[t]]$ N large enough]

The center $Z(\hat{\mathfrak{g}})$ of $\tilde{U}_{h^\vee}(\hat{\mathfrak{g}})$ is canonically isomorphic to the algebra $\mathrm{Fun}[Op_{{}^L\mathfrak{g}}(D^\times)]$

${}^LG$-oper on X: principal ${}^LG$ bundle + connection + reduction to Borel with some condition

eg sl_2 $\quad \nabla = \partial_t + \begin{pmatrix} \cdot & q(t) \\ 1 & 0 \end{pmatrix}$ $\quad q(t) = \sum q_n t^n \in \mathbb{C}((t))$

$\longleftrightarrow$ $\partial^2 - q(t)$ Sturm-Liouville operator " $\sum S_m t^{-m-2}$

$Z(\hat{sl}_2) \simeq \mathbb{C}[q_n]_{n\in\mathbb{Z}} \simeq \mathbb{C}[S_m]$ Sugawara operators

Let $V(\hat{\mathfrak{g}}) = \mathrm{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}k}^{\hat{\mathfrak{g}}} \mathbb{C}_{-h^\vee}$

$Z(\hat{\mathfrak{g}}) \longrightarrow \mathrm{End}_{\hat{\mathfrak{g}}} V(\hat{\mathfrak{g}}) = \mathfrak{z}(\hat{\mathfrak{g}})$

Theorem 2 This map is surjective & image is Fun $\mathrm{Op}_{^LG}(D)$ ... e.g. sl_2 $\mathrm{End} = \mathbb{C}[S_n]_{n \geq 0} = \mathbb{C}[S_m]_{m \leq -2}$ Sugawara

Given $\rho \in \mathrm{Op}_{^LG}(D) \Rightarrow \hat{\mathfrak{g}}$-module
$V_\rho = V(\hat{\mathfrak{g}}) / \{\mathrm{Ker}\, \bar{\rho} : \mathfrak{z}(\hat{\mathfrak{g}}) \to \mathbb{C}\}$

Theorem 3 V_ρ is irreducible, $\neq 0$ & these are all irreducible unramified $\hat{\mathfrak{g}}_{-h^\vee}$-modules.

Theorem (Beilinson-Drinfeld) 1. If ρ extends to $\tilde{\rho} \in \mathrm{Op}_{^LG}(X)$ Then $\Delta(V_\rho) \neq 0$ on Bun_G, get Hecke eigensheaf corresponding to underlying $(\mathcal{F}, \nabla)$ G-local system. $\mathrm{Op}_{^LG}(X) \hookrightarrow \mathcal{L}oc_{^LG}(X)$
2. Otherwise $\Delta(V_\rho) = 0$. indep of $x \in X$.

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$\hat{\mathfrak{g}}$ "critical" level extension of $\mathfrak{g} \otimes \mathbb{C}((t))$

Smooth $\hat{\mathfrak{g}}$-modules M: $\forall v \in M \; \exists N \geq 0$ s.t. $\mathfrak{g} \otimes t^N \mathbb{C}[[t]] \cdot v = 0$

& $\mathbb{K}$ acts as identity (critical level)

$U_1(\hat{\mathfrak{g}}) = U\hat{\mathfrak{g}}/\mathbb{K}-1$, $\tilde{U}(\hat{\mathfrak{g}}) = \varprojlim U_1(\hat{\mathfrak{g}})/U_1(\hat{\mathfrak{g}}) \cdot \mathfrak{g} \otimes t^N \mathbb{C}[[t]]$

$Z(\hat{\mathfrak{g}})$ center of $\tilde{U}(\hat{\mathfrak{g}})$

Theorem (Feigin-Frenkel) $Z(\hat{\mathfrak{g}}) \simeq \mathrm{Fun}\, \mathrm{Op}_{^LG} D^*$ "^LG-opers on D^*"

(G simply conn, ^LG adjoint type)

Consider categories of Harish-Chandra modules for $(\hat{\mathfrak{g}}, K)$ $K \subset G[[t]]$, & relate them/their derived categories to categories of coherent sheaves on varieties associated to ^LG.

Simplest case $K = G[[t]] = G(\mathcal{O})$. Example of a $(\hat{\mathfrak{g}}, G(\mathcal{O}))$-module: vacuum module $V(\mathfrak{g}) = V_\kappa(\mathfrak{g}) = \mathrm{Ind}^{\hat{\mathfrak{g}}}_{\mathfrak{g}[[t]] \oplus \mathbb{C}K} \mathbb{C}$.

Last time: $\mathrm{End}_{\hat{\mathfrak{g}}} V(\mathfrak{g}) \cong \mathrm{Fun}\, \mathrm{Op}_{^LG}(D)$.

$$\begin{array}{ccc} \mathrm{End}_{\hat{\mathfrak{g}}} V(\mathfrak{g}) & \cong & \mathrm{Fun}\, \mathrm{Op}_{^LG}(D) \\ \uparrow & & \uparrow \text{restriction} \\ Z(\hat{\mathfrak{g}}) & \simeq & \mathrm{Fun}(\mathrm{Op}_{^LG}(D^*)) \end{array}$$

$\mathcal{C}_0$: category of $(\hat{\mathfrak{g}}, G(\mathcal{O}))$-modules [$\mathbb{K} = \mathrm{Id}$] s.t. $Z(\hat{\mathfrak{g}})$ action factors through $\mathfrak{Z}(\hat{\mathfrak{g}}) := \mathrm{End}_{\hat{\mathfrak{g}}}(V(\mathfrak{g}))$

$\mathcal{M}_0$: category of q-coh sheaves on $\mathrm{Op}_{^LG}(D) = \mathfrak{Z}(\hat{\mathfrak{g}})$-mod.

$F_0 : \mathcal{C}_0 \rightleftarrows \mathcal{M}_0 : G_0$ $\quad F_0 : M \longmapsto \mathrm{Hom}_{\hat{\mathfrak{g}}}(V(\mathfrak{g}), M)$

$G_0 : F \longmapsto V(\mathfrak{g}) \otimes_{\mathfrak{Z}(\hat{\mathfrak{g}})} F$

Theorem F_0, G_0 are mutually inverse equivalences of categories

More generally any $(\hat{\mathfrak{g}}, G(\mathcal{O}))$ module is supported in $\mathrm{Spec}\, Z(\hat{\mathfrak{g}})$ on the union for every $\lambda \in \mathrm{Irr}\, ^LG$ of the formal nbhds of space of opers with regular singularity with residue λ:

i.e. $(\hat{\mathfrak{g}}, G(\mathcal{O}))$-modules decomposes into blocks labelled by $\lambda \in \mathrm{Irr}\, ^LG$. We have only consider one supported precisely on $\mathrm{Op}(D)$, correspondence doesn't quite extend to full block.

Steps in proof:

1. $V(\mathfrak{g})$ free over $\mathfrak{z}(\hat{\mathfrak{g}})$ — analog of Kostant's theorem that $U\mathfrak{g}$ is free over $Z\mathfrak{g}$.

Proof: first prove for associated gradeds,

$\mathrm{Sym}\, \mathfrak{g}(K)/\mathfrak{g}(\mathcal{O}) = \mathrm{Fun}\, \mathfrak{g}^*[[t]]dt$

$\mathrm{gr}\, \mathfrak{z}(\hat{\mathfrak{g}}) = \mathrm{Fun}(\mathfrak{g}^*[[t]]dt)^{G(\mathcal{O})}$ (by F-F theorem)

Kostant proof in f.d. case: nilpotent cone (zero locus) is complete intersection ⇝ irred (& reduced). (Nilcone is zero locus of polynomials).

So look at $\mathrm{Fun}\, \mathfrak{g}[[t]] / (\mathrm{Fun}\, \mathfrak{g}[[t]]^{G[[t]]})_+ = \mathrm{Fun}\, J\mathcal{N}$

jet scheme of nilpotent cone

— jet schemes often reducible, nonreduced etc!

<u>Theorem</u> $J_n\mathcal{N}$ are irr. reduced, complete intersection

... since $\mathcal{N}$ has rational singularities (<u>Mustata</u>)

— jet schemes of irred reduced schemes are also irred, red iff at most rational (canonical!) singularities (using motivic integration).

2. $V(\mathfrak{g})$ projective object in $\mathcal{C}_0$ .... Drinfeld

$\mathrm{Hom}_{\hat{\mathfrak{g}}}(V(\mathfrak{g}), V(\mathfrak{g})) = \mathrm{Fun}\, \mathrm{Op}_{{}^LG}(D)$

What is $R^i\,\mathrm{Hom}_{\hat{\mathfrak{g}}}(V(\mathfrak{g}), V(\mathfrak{g}))$ in full category of $\hat{\mathfrak{g}}$-modules?

Answer: $\Omega^i(\mathrm{Op}_{{}^LG}(D))$ Frenkel-Teleman

... ext algebra is de Rham algebra....

can calculate exts

Classical picture	geometric picture
V $G(F)$-module $F = \mathbb{F}_q((t))$	Cat of smooth $\hat{\mathfrak{g}}$-modules: $G(K)$ acts on <u>category</u>, not on the modules!
V^K-module over $\mathcal{H}(G,K)$ Hecke algebra	Category $\mathcal{C}(\hat{\mathfrak{g}},K)$ of $(\hat{\mathfrak{g}},K)$-modules, action of $\mathcal{H}(G,K)$ = category of K-equiv perverse sheaves on $G((t))/K$ (derived) - with monoidal structure. - in general convolution hard to define; have nonproper maps involved at least for K large enough.

Spherical Hecke category $\mathcal{H}_{sph} = \mathcal{H}(G((t)), G[[t]])$ - $G[[t]]$-equivariant perverse sheaves on $Gr = G((t))/G[[t]]$

Theorem (Mirkovic-Vilonen) $\mathcal{H}_{sph} \simeq Rep\, {}^L G$

$\mathcal{C}_0 \curvearrowleft$ $\curvearrowleft \mathcal{M}_0$

Recall $\mathcal{M}_0$ - q-coh sheaves on $Op_{{}^L\mathfrak{g}}(D)$, have universal bundle $\mathcal{T} \to Op_{{}^L\mathfrak{g}}(D) \times D$

$\mathcal{T}_0 \to Op_{{}^L\mathfrak{g}}(D)$ fiber at zero bundle. W ${}^L G$-module $\Rightarrow W_{\mathcal{T}_0} = \mathcal{T}_0 \times_{{}^L G} W$

Vector bundle on $Op_{{}^L G}(D) \Rightarrow$ action of $Rep\, {}^L G$ on $\mathcal{M}_0$ $W * \mathcal{F} = W_{\mathcal{T}} \otimes \mathcal{F}$

Theorem $\mathcal{C}_0 \simeq \mathcal{M}_0$ is $\mathcal{H}_{sph}$-equivalent.

$\mathcal{C}_0$ $\mathcal{M}_0$

$V(\mathfrak{g}) \rightsquigarrow \mathcal{O}_{Op_{{}^L G}(D)}$

Corollary $W \in Ob\, Rep\, {}^L G$, IC_W corresponding spherical sheaf

Then $IC_W * V(\mathfrak{g}) = V(\mathfrak{g}) \otimes_{\mathfrak{Z}(\mathfrak{g})} W_{\mathcal{T}} \simeq V(\mathfrak{g}) \otimes W$ as vector space

ie $V(\mathfrak{g})$ is eigenobject of spherical Hecke algebra!

M $(\mathfrak{g}, K)$-module. $G \to G/K$ $\mathcal{M} = G \times_K M$ twist.

Given $\mathcal{F} \in D\text{-mod}(G/K) \Rightarrow H^0_{DR}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{M})$ is a $(\mathfrak{g}, K)$-module

$\Rightarrow$ action. eg. $M = \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}} \mathbb{C} \Rightarrow \mathcal{F} * M = \Gamma(G/K, \mathcal{F})$

This is categorization of action of G on category of reps - take δ-fns at $g \in G$, action on category of reps will be action of g.

X smooth projective curve, E ${}^L G$-local system $\Rightarrow$ assign Aut_E D-mod on Bun_G which is E-Hecke eigensheaf.

$H_W: D(Bun_G) \to D(X \times Bun_G)$ $W \in Rep\, {}^L G$

Hecke eigensheaf! $H_W * Aut_E \xrightarrow{\sim} W_E \boxtimes Aut_E$

We've constructed eigenobject $V(\mathfrak{g})$ in categorical world,

... use localization functor $\Delta : (\mathfrak{g}, G[[t]])\text{-mod} \longrightarrow$ (twisted) $\mathcal{D}$-mod on Bun_G

This intertwines $\mathcal{H}_{sph}$ actions!

$V(\mathfrak{g}) \to$ Hecke eigensheaf ...

$\rho \in Op_{^LG}(D) \Rightarrow V_\rho = V(\mathfrak{g}) / \text{Ideal corresponding to } \rho.$

$\Delta V_\rho \neq 0$ iff ρ comes from global opers,

get Hecke eigensheaf with eigenvalue ρ. (B–D)

~~Rem~~

Noncritical level: won't have Hecke eigensheaves! e.g. vacuum itself.. can act on it by spherical Hecke algebras reps we get won't be related.

Now replace $G[[t]]$ by Iwahori subgroup I

$\mathcal{C}$ category of $(\hat{\mathfrak{g}}, I)$-modules which live over "nilpotent "G-opers" $Op_{^LG}(D) \subset n\,Op_{^LG} \subset Op_{^LG}(D^\times)$

$\mathfrak{g} = sl_2 \quad Op_{sl_2} = \{(\mathcal{F}, \nabla, \mathcal{F}_B)\} \quad \partial_t + \begin{pmatrix} * & * \\ 1 & * \end{pmatrix} \sim \partial_t + \begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix}$

$\sim \partial_t^2 - v$

Regular opers: $v \in \mathbb{C}[[t]]$

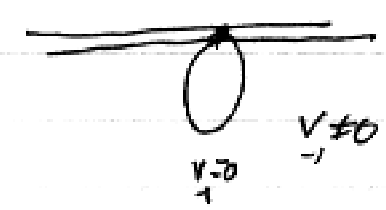
Nilpotent opers: $v \in t^{-1}\mathbb{C}[[t]] \longrightarrow v_{-1}$ "residue" of oper

In general: $n\,Op_{^LG} \longrightarrow {}^L\mathfrak{n} / Ad\, {}^LB \qquad 0$ fiber $=$ regular opers

$\mathcal{M}$ = category of quasicoh sheaves on nilpotent Miura opers

$$\begin{array}{ccccl} \text{Spring}(Res\,\rho) \subset n\,M\,Op_{^LG} & \longrightarrow & \widetilde{{}^L\mathfrak{n}} / Ad\, {}^LB & & \text{Springer resolution} \\ \downarrow \qquad\qquad \downarrow & & \downarrow & & \\ \rho \in n\,Op_{^LG} & \xrightarrow{Res} & {}^L\mathfrak{n} / Ad\, {}^LB & & \end{array}$$

$Sl_2: \; v_{-1} = 0$ glue in $\mathbb{P}^1$, $v_{-1} \neq 0$ glue in point (doubled)



Singular quasiprojective variety!

Conjecture (Frenkel–Gaitsgory) $D^b(\mathcal{C}) \simeq D^b(\mathcal{M})$

as categories "over" $n\,Op_{^LG}$.

--- really not I-monodromic modules

i.e. $\mathcal{C}$ fibers over space of opers with fibers category of sheaves on Springer fiber. Should be compatible with

Hecke action: $\mathcal{H} \in (G((t)) / I)_{mon} \circlearrowleft \mathcal{D}(\mathcal{C})$

Bezrukavnikov $\cong$

$D^b_{^LG}(\mathcal{O}_Z\text{-mod}) \circlearrowleft D^b(\mathcal{M})$

Z = Steinberg (or dg version)