N. Ganter - Orbifold genera, K(n)-local spectra, product formulas & power operations   Toronto 12/09

Dijkgraaf - Moore - Verlinde - Verlinde: (elliptic cohomology)

$$\sum_{n\geq 0} \phi_{ell,orb}(M^n/\!/\Sigma_n)\, t^n = \prod_{\substack{m\geq 1\\ n\geq 0\\ \ell}} \left(\frac{1}{1-t^m q^n y^\ell}\right)^{c(mn,\ell)}$$

where $\phi_{ell}(M) = \sum_{\substack{n\geq 0\\ \ell\in\mathbb{Z}}} c(n,\ell)\, q^n y^\ell$

$$RHS = \exp\Big(\sum_{n\geq 1} V_n(\phi(M))\, t^n\Big)$$

$\downarrow$ Borcherds.

V_n : Hecke operators on Jacobi forms ($T(n)$)

- Borcherds lift of "Jacobi" form $\phi_{ell}(M)$

Todd genus formula: (K-theory)
"topological Euler characteristic" Td_{top} (topological Td)

$$\sum_{n\geq 0} Td_{top}(M^n/\!/\Sigma_n)\, t^n = \left(\frac{1}{1-t}\right)^{Td\, M}$$

Goal: • systematic understanding of such formulas from POV of stable homotopy theory; h'py theoretic refinement; higher chromatic versions
• "K(n)-local category as natural habitat for orbifold genera"

LHS: involves $M \longmapsto \sum_{n\geq 0} M^n \circlearrowleft \Sigma_n$
total power operation in $MU(pt)$

RHS: only depends on $\phi_{ell}(M)$ / $Td(M)$: function of ϕ, takes sums into products (like total power operations!)

genus: map of spectra (applied to $MU^*(pt)$)
$MU \xrightarrow[Td]{} K$ (Conner-Floyd) $MU \longrightarrow Ell$

RHS of Borcherds formula: similar to inverse of Rezk logarithm, see Hecke operators as cohomology operations

Td_{top} : chromatic level 1 phenomenon

ϕ_{ell} : " " 2 "

Q: Does DMVV formula just reflect fact that a map of spectra preserves power operations (i.e. H_∞-map)

A: "Yes"

Setup Pick $Ell = E_2$ second Morava E-theory

Theorem (G.) Any H_∞-map $\phi : MU \to E_n$ has a formula

$$(*) \quad \sum_{n \geq 0} \phi_{orb}(M^n/\Sigma_n)\, t^n = \exp\Big(\sum_{k \geq 0} t^{p^k} T(p^k)\big(\phi(M)\big)\Big)$$

$T(p^k)$ ... (p-typical) Hecke operators in E_n-theory

Why E_2?
- elliptic spectrum
- Borel equivariant versus $E_2(EG \times_G -)$ are well understood
- H_∞-spectrum, well understood
- Hecke operators
- more reasons later

Homotopy-theoretic refinement: $(*)$ comes from an equality of cohomology operations

$$S_t(x) = \exp\Big(\sum_{k \geq 0} t^{p^k} T(p^k)(x)\Big)$$

total symmetric power in E_n.

Recover $(*)$ in case of a point.

How to define orbifold versions?

$$\mathrm{Td}_{top}(M \circlearrowleft \underset{\text{finite}}{G}) = \frac{1}{|G|}\sum_{g\in G} \mathrm{Trace}(g \,|\, \underbrace{\mathrm{Td}_G(M)}_{\in K_G(pt) = R(G)})$$

Atiyah-Segal: $\mathrm{Tr}(g\,|\,\mathrm{Td}_G(M)) = \int_{M^g} td(M^g) \frac{1}{\prod_{\xi}\prod_i (1-\xi e^{x_i(N^g_\xi)})}$

Chern roots of eigenbundles of $N^g \circlearrowleft g$ normal bundle to fixed pt loci

Ando-French: $\phi: MU \longrightarrow E_n$

$K_G(-) \implies E_G(-) := E(EG \times_G -)$

$$\begin{tikzcd}
K_G(pt) \arrow[d, "\mathrm{Tr}(g|-)"'] \arrow[r, Rightarrow] & E(BG) \arrow[r, "\mathrm{eval}_\alpha", "\text{Hopkins-Kuhn-Ravenel}"'] & D \\
\mathbb{C} & &
\end{tikzcd}$$

evaluate at $\alpha = (g_1, \dots g_n)$ n-tuple of commuting elements of p-power order

Def (Ando-French): $\phi: MU \longrightarrow E_n$, $\phi_{orb}(M \circlearrowleft G) := \frac{1}{|G|}\sum_\alpha \mathrm{eval}_\alpha(\phi_G(M))$

(ϕ_G = Borel equivariant version of ϕ)

Results

- "integrality" $\phi_{orb}(M \circlearrowleft G) \in E_n^0$ (ie don't need to invert $\frac{1}{|G|}$)

- ϕ_{orb} is a well defined notion of orbifolds ...
  ie if $M/\!/G \simeq N/\!/H$ as orbifolds $\implies$
  $\phi_{orb}(M \circlearrowleft G) = \phi_{orb}(N \circlearrowleft H)$

Integrality: in K-theory: $R(G) \longrightarrow \mathbb{Z}$

$$\frac{1}{|G|}\sum_G \mathrm{Tr}(g|-) = \langle -, \mathbb{1} \rangle_G$$

Strickland inner products

$BG_+ \wedge BG_+ \longrightarrow BG_+$

$BG_+ \xrightarrow[B(p_G)_+]{} S^0 = B1_+$ $\qquad p_G: G \to 1$

In $K(n)$-local category:

let $B = B(\rho_G)_+ \circ \mu$

$$b^{\#} : BG_+ \xrightarrow[\underset{\text{Strickland}}{\cong}]{} DBG_+ \quad (K(n) \text{ local Spanier-Whitehead dual})$$

So conclude, get $S^0 \xrightarrow{\eta} BG_+$, $BG_+ \xrightarrow{B\delta_+} BG_+ \wedge BG_+$

(only exists $K(n)$ locally)

$\Rightarrow$ get inner product $E(BG) \otimes E(BG) \to E^0$

inner product with 1 is $\eta^* : E(BG) \to E^0$

<u>Prop.</u> $\chi, \xi \in E(BG)$

$$\langle \chi, \xi \rangle = \frac{1}{|G|} \sum_{\substack{\alpha \\ n\text{-tuples}}} \chi(\alpha)\, \xi(\alpha)$$

Corollary: $\frac{1}{|G|} \sum \mathrm{ev}_\alpha$ is η^* (inner product w/ 1) $\Longrightarrow$ integrality;

<u>Orbifolds</u>

$\phi_G(M)$ is pushforward of $1 \in E_G^0(M)$

$$\begin{tikzcd}
E(EG \times_G M) = E_G(M) \arrow[r, "\text{Thom isom}", "\Theta_\phi"'] & E_G^{-d}(M^{-\tau}) \arrow[d, "P\text{-}T", "\text{Pontrjagin-Thom collapse}"'] \\
E^{-d}(BG) \arrow[d, "\eta^*"] & E^{-d}(S) \arrow[l, phantom, "="] \\
E^{-d} \arrow[r, phantom, "\longleftrightarrow"] & D \circ \mathrm{Borel}(\rho_G)_+
\end{tikzcd}$$

$K(n)$ local dual

P-T is Spanier Whitehead dual of

$$\begin{tikzcd}
\pi : M /\!/ G \arrow[r, "\pi"] \arrow[dr, "\text{well defined map of orbifolds}"'] & pt /\!/ G \arrow[d] \\
& pt /\!/ 1
\end{tikzcd}$$

<u>Theorem</u> (G) G finite, X finite G-CW complex have functor (of X) isomorphism $\mathrm{Borel} \circ D_G \simeq D \circ \mathrm{Borel}$ in $K(n)$ local category (consequence of Strickland)