V. Ginzburg - Harish-Chandra Homomorphisms & Calogero-Moser 10/30/00

$\mathfrak{g}$ s.s. Lie alg / $\mathbb{C}$ G adjoint group of $\mathfrak{g}$

$\mathfrak{h}$ Cartan T max. torus, Lie $T = \mathfrak{h}$

W = Weyl group $U\mathfrak{g} \supset Z(U\mathfrak{g})$

Harish-Chandra: $Z(U\mathfrak{g}) \xrightarrow{\sim} (\mathrm{Sym}\,\mathfrak{h})^W$

Let $\mathcal{D}(X)$ = algebraic regular diffops on X smooth

$Z(U\mathfrak{g}) = \mathcal{D}(G)^{G\times G}$ bi-invariant diffops on G.

$S\mathfrak{h}^W = T \rtimes W$-invariant diffops on T (i.e. constant coefficient) $\}$ H-C

1964 More general: $\mathcal{D}(G)^{G\times G} \hookrightarrow \mathcal{D}(G)^{\mathrm{Ad}\,G}$,

map to $\mathcal{D}(T)^W$ (no longer T-invariant)

$\Phi : \mathcal{D}(G)^{\mathrm{ad}\,G} \longrightarrow \mathcal{D}(T)^W$ not iso (LHS much larger)

"rational case" - replace groups by Lie algebras

$\phi : \mathcal{D}(\mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h})^W$ no longer isom.

Both sides are filtered by order of diffops, map preserves filtration. $\mathrm{gr}\,\phi : \mathbb{C}[\mathfrak{g}\oplus\mathfrak{g}]^{\mathrm{ad}\,\mathfrak{g}} \longrightarrow \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}]^W$

- just restriction - "double analog" of Chevalley restriction map. ($\mathrm{gr}\,\mathcal{D}(X) = \mathbb{C}[T^*X]$)

Radial part construction $\mathfrak{g}^{rs}$ regular semisimple $\subset \mathfrak{g}$ open dense

$\mathfrak{h}^{reg} = \mathfrak{h} \cap \mathfrak{g}^{rs}$

$G/T \times \mathfrak{h}^{reg} \longrightarrow \mathfrak{g}^{rs}$ $(g,h) \longmapsto \mathrm{Ad}\,g \cdot h$

W-Galois covering.

Given $u \in \mathcal{D}(\mathfrak{g}^{rs})^{\mathrm{ad}\,\mathfrak{g}}$ $\rightsquigarrow$ regard as

$\tilde{u} \in \mathcal{D}(G/T \times \mathfrak{h}^{reg})^G$ on étale cover, lift diffop, invariant under left action $G \circlearrowright G/T$.

In particular $\tilde{u}$ acts on $\mathbb{C}[G/T \times \mathfrak{h}^{reg}]^G = \mathbb{C}[\mathfrak{h}^{reg}]$ by a differential operator

$\Rightarrow$ $\mathcal{D}(G/T \times \mathfrak{h}^{reg})^G \longrightarrow \mathcal{D}(\mathfrak{h}^{reg})$

$\mathcal{D}(\mathfrak{g}^{rs})^{\mathrm{ad}\,\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h}^{reg})^W$

$u \longmapsto u^{\mathrm{radial}}$

R = root system of $(\mathfrak{g}, \mathfrak{h})$, R_+ positive roots,

$\delta = \prod_{\alpha \in R_+} \alpha$ skew polynomial on $\mathfrak{h}$ $f \in \mathbb{C}[\mathfrak{h}]$

$\phi : u \longmapsto \frac{1}{\delta} \circ u^{radial} \circ \delta$

which acts as $\mathbb{C}[\mathfrak{h}^{reg}] \ni f \longmapsto \frac{1}{\delta} u^{radial} \cdot (\delta f)$

– δ never vanishes on $\mathfrak{h}^{reg}$.

(δ cuts out irregular part).

[HC proved that $\phi(u) \in \mathcal{D}(\mathfrak{h})^W$ i.e. not just on $\mathfrak{h}^{reg}$, provided u is regular on $\mathfrak{g}$ not just $\mathfrak{g}^{rs}$

$u \in \mathcal{D}(\mathfrak{g}^{rs})$ --- not true for just radial part.

[We won't need this extension $\mathfrak{h}^{rs} \rightsquigarrow \mathfrak{h}$].

B Borel $\subset G$ assoc to R_+ : replace Cohen's covering by

$p : G \times_B \mathfrak{b} \longrightarrow \mathfrak{g}$, δ occurs as Jacobian of this map p.]

Let V = f.d. G-module, irreducible (recall G adjoint $\Rightarrow$ h.w. in root lattice) $\rightsquigarrow$ 0 weight space $V\langle 0\rangle = V^T = V^{\mathfrak{h}} \neq 0$.

Frobenius reciprocity! $\mathbb{C}[G/T] = \bigoplus_{\substack{\text{simple} \\ G\text{-module } V}} V^* \otimes V\langle 0\rangle$ (Peter-Weyl)

If $u \in \mathcal{D}(\mathfrak{g}^{rs})^{ad\,\mathfrak{g}}$, u action on $\mathbb{C}[\mathfrak{g}^{rs}]$ commutes with adjoint action of $\mathfrak{g}$.

– can take any isotypic component in $\mathbb{C}[\mathfrak{g}^{rs}]$, not just invariant part as before

$\rightarrow$ $\mathcal{D}(\mathfrak{g}^{rs})^{ad\,\mathfrak{g}}$ acts on $\mathrm{Hom}_{\mathfrak{g}}(V^*, \mathbb{C}[G/T \times \mathfrak{h}^{reg}])$

$= \mathbb{C}[\mathfrak{h}^{reg}] \otimes V\langle 0\rangle$

So get a map $\tilde{\Psi}_V : \mathcal{D}(\mathfrak{g}^{rs})^{ad\,\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h}^{reg}, \mathrm{End}_{\mathbb{C}} V\langle 0\rangle)^W$

differops with values in $\mathrm{End}_{\mathbb{C}} V\langle 0\rangle$

[$\mathcal{D}(X, A) := \mathcal{D}(X) \otimes A$ A-valued diffops]

$$\Psi_V := \frac{1}{\delta} \circ \tilde{\Psi}_V \circ \delta : \mathcal{D}(\mathfrak{g}^{rs})^{ad\,\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h}^{reg}, \mathrm{End}_{\mathbb{C}} V\langle 0\rangle)^W$$

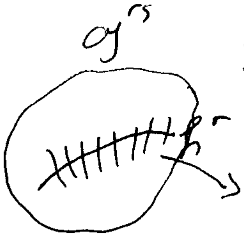
- $U\mathfrak{g}^{ad\,\mathfrak{h}}$ centralizer of $\mathfrak{h} \subset U\mathfrak{g}$, contains $\mathfrak{h}$.
- $(U\mathfrak{g})_{\mathfrak{h}} = U\mathfrak{g}^{ad\,\mathfrak{h}} / \mathfrak{h}(U\mathfrak{g})^{ad\,\mathfrak{h}}$ ($\mathfrak{h}$ central $\rightsquigarrow$ generates 2-sided ideal)

$(U\mathfrak{g})^{ad\,\mathfrak{h}}$ acts on any weight space $V\langle\mu\rangle$ of a $\mathfrak{g}$-module V.

$\Rightarrow$ $(U\mathfrak{g})_{\mathfrak{h}}$ acts on $V\langle 0\rangle$ for any V. $(U\mathfrak{g})_{\mathfrak{h}} \xrightarrow{\omega} \mathrm{End}_{\mathbb{C}} V\langle 0\rangle$

" endomorphism ring of functor $V \longmapsto V\langle 0\rangle$ "

Theorem $\exists!$ algebra homomorphism $\tilde{\Psi} : \mathcal{D}(\mathfrak{g}^{rs})^{\mathrm{ad}\,\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h}^{reg}, (U\mathfrak{g})_{\mathfrak{h}})^W$
s.t. for any $\mathfrak{g}$-module V, and $u^{\mathfrak{g}}$,
$\omega \circ \tilde{\Psi}(u) = \tilde{\Psi}_V(u)$. $\hookrightarrow$ (fin. dim. - we used decomp of G/T)



__Sketch of proof__

infi nbhd of $\mathfrak{h}^r < \mathfrak{g}^{rs}$ (formal completion)
$\hat{\mathfrak{g}}_{\mathfrak{h}}$ $\parallel$ For __any__ $\mathfrak{g}$-module V with $\mathfrak{h}$ diagonalizable
(or just locally finite),
V-valued functions $\mathbb{C}[\hat{\mathfrak{g}}_{\mathfrak{h}}, V]^{\mathrm{ad}\,\mathfrak{g}} \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^{reg}, V\langle 0\rangle]$
--- $\mathfrak{g}$ action transverse to $\mathfrak{h}^r$ ---
looks like $\widehat{G/T}_1$ - infinitesimal Frobenius reciprocity

(our W-cover is infinitesimally isomorphism). ($\hat{\mathfrak{g}}_{\mathfrak{h}} \sim \widehat{G/T} \times \mathfrak{h}^{reg}$..)
Apply our previous construction to
$V = U\mathfrak{g} \otimes_{(U\mathfrak{g})^{\mathrm{ad}\,\mathfrak{h}}} (U\mathfrak{g})_{\mathfrak{h}}$ induced $U\mathfrak{g}$ module with diagonalizable $\mathfrak{h}$-action.

- a cyclic $U\mathfrak{g}$-module, $V = U\mathfrak{g} / U\mathfrak{g}\cdot\mathfrak{h}$
$V\langle 0\rangle = 1 \otimes (U\mathfrak{g})_{\mathfrak{h}}$, so universal object is itself a $V\langle 0\rangle$. ∎

__Drinfeld__ $\mathfrak{g}^{rs} \xleftarrow{W} G/T \times \mathfrak{h}^r$, étale $\Rightarrow$ get map $\mathcal{D}(\mathfrak{g}^{rs})^G = (\mathcal{D}(G/T \times \mathfrak{h}^r)^G)^W$

$\mathcal{D}(G/T \times \mathfrak{h}^r) = \mathcal{D}(G/T) \otimes \mathcal{D}(\mathfrak{h}^{reg})$
$\mathcal{D}(G/T)^G = (U\mathfrak{g})_{\mathfrak{h}}$ - defined by value at $1T$
$(\mathcal{D}(G/T) \otimes \mathcal{D}(\mathfrak{h}^r))^G = \mathcal{D}(G/T)^G \otimes \mathcal{D}(\mathfrak{h}^r) = (U\mathfrak{g})_{\mathfrak{h}} \otimes \mathcal{D}(\mathfrak{h}^{rs})$

Recall $\Psi = \frac{1}{\delta} \circ \tilde{\Psi} \circ \delta$. Let $\{e_\alpha, \alpha \in R\}$ Chevalley basis in $\mathfrak{g} \ominus \mathfrak{h}$

Consider constant coefficient Laplacian $\Delta_{\mathfrak{g}}$ on $\mathfrak{g}$ $= \sum \frac{\partial^2}{\partial x_i^2}$ in ortho basis.

$$\boxed{\text{What is } \Psi(\Delta_{\mathfrak{g}})\,? = \Delta_{\mathfrak{h}} + \sum_{\alpha \in R} \frac{e_\alpha e_{-\alpha}}{\alpha^2}}$$
(HC map $\phi(\Delta_{\mathfrak{g}}) = \Delta_{\mathfrak{h}}$)

in $(U\mathfrak{g})_{\mathfrak{h}}$, 0-th order diff op

Order of diff op is preserved, symbol should be same as Laplacian — δ kills first order term
so get $\Delta_{\mathfrak{h}}$ + function

Calogero-Moser

	Classical	Quantum
gl_n	$H = \sum p_i^2 + c \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}$	$\Delta + \underbrace{k(k+1)}_{\text{Planck constant}} \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}$, $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$
$\mathfrak{g}$	$H = \lvert p \rvert^2 + \sum_{\alpha \in R} \frac{c_\alpha (c_\alpha + 1)}{\alpha^2(x)}$	$\Delta - \sum_{\alpha \in R} \frac{c_\alpha (c_\alpha + 1)}{\alpha^2}$

Specify a function $c : R \to \mathbb{C}$ which is W-invariant
-- either 1 or 2 orbits of W, so just 1 or 2 numbers for any simple
$\alpha \mapsto c_\alpha$ $\quad R = A_n$ only 1 number

So $\psi(\Delta_{\mathfrak{g}})$ looks like quantum Calogero-Moser with constant replaced by $e_\alpha e_{-\alpha} \in (U\mathfrak{g})_{\mathfrak{h}}$ -- "correct" C-M operator.
Radial part on group gives $\sin^2$ denominator
— $\psi(\Delta_{\mathfrak{g}})$ is quantum spin Calogero-Moser Hamiltonian.

From now on $\mathfrak{g} = \mathfrak{sl}_n$.
$(U\mathfrak{g})_{\mathfrak{h}} \twoheadrightarrow V\langle 0 \rangle$. look for V with $\dim V\langle 0 \rangle = 1$
$\Rightarrow$ get scalar diffops from $\psi(\Delta_{\mathfrak{g}})$
-- don't have many such for general Lie algebras which don't work. — quantum version of coadjoint $\mathcal{O}_{\min}$.

Construction of V_k: (corresponding to $k(k+1)$)
As a space $V_k = (x_1 \cdots x_n)^k \, \mathbb{C}[x_1^{\pm 1}, \ldots x_n^{\pm 1}]_{\text{total deg}=0}$

$$\left(\ 1\ \right) \quad E_{ij} \text{ matrix units acts as } x_i \frac{\partial}{\partial x_j} - k \cdot \delta_{ij} \quad \text{(Kronecker } \delta\text{)}$$

$\Rightarrow$ $V\langle 0 \rangle = \mathbb{C} \cdot \left((x_1 \cdots x_n)^k \cdot 1 \right)$ 1-dimensional.
- correction $k \cdot \delta_{ij}$ came to make this degree zero
— need as many variables as dimension of Cartan, to get $V\langle 0 \rangle$ 1 dim.
For V f.d.: take $S^{kn}(\mathbb{C}^n)$, — but we want continuous parameter.

Let L = canonical bundle on $\mathbb{CP}^{n-1}$ with 0-section removed.

$$\begin{array}{c} L \\ \downarrow \pi \\ \mathbb{CP}^{n-1} \end{array} \qquad T \subset GL_n \text{ has unique open orbit } U \subset \mathbb{CP}^{n-1}$$

L_U Twisted differential operators: $\mathcal{D}(L_U)^{\mathbb{C}^*}/(Eu-k) = \mathcal{D}_k$

$\downarrow \mathbb{C}^*$ generated by Euler field Eu.

U

GL_n acts on L, $\mathfrak{a}: \mathfrak{gl}_n \longrightarrow \mathcal{D}(L_U)^{Eu}$

<u>Prop</u> $V_k = \mathcal{D}_k / \mathcal{D}_k \cdot \mathfrak{a}(\mathfrak{h})$

[Rapoport $U \sim$ Drinfeld UHS for field of 1 elt, which is affine torus etc
- doesn't occur for other groups
- lines which don't lie on rational hyperplanes
hyperplanes coming from a fixed basis $\sim$ field $\mathbb{F}_1$.]

<u>Notation</u> $\phi_k = \Psi_{V_k}$

<u>Claim</u>: $\phi_0 = \phi$ Harish-Chandra (valid in twisted setup!)

So $\phi_k(\Delta_{\mathfrak{g}}) = \Delta_{\mathfrak{h}} - k(k+1)\sum_{i\neq j}\frac{1}{(x_i-x_j)^2} =: L_k$

$(\mathfrak{g}=\mathfrak{gl}_n)$ $\phi_k : \mathcal{D}(\mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h}^{\mathrm{reg}})^W$

What are kernel & image of this map?

Let $\mathcal{C}_k$ = centralizer of L_k in $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}})^W$: quantum integrals of motion

<u>Theorem (Opdam)</u> There is an algebra isomorphism $\sigma : S\mathfrak{h}^W \xrightarrow{\sim} \mathcal{C}_k$ with Casimir $\Delta \longmapsto L_k$, with property that principal symbol of $\sigma(p)$ is p. (gives identity on symbols).

constant coeff diffops

<u>Def</u> $\mathcal{B}_k$ is subalgebra in $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}})^W$ generated by $\mathcal{C}_k$ and $\mathbb{C}[\mathfrak{h}]^W$. (as 0-order operators)

<u>Theorem</u> $\mathrm{Im}(\phi_k) = \mathcal{B}_k \quad \forall k$.

Classical H-C: what is $\mathrm{Im}(\phi=\phi_0)$? $= \mathcal{D}(\mathfrak{h})^W$
- highly nontrivial, - proved in last five years.

Asked first by Wallach, proved for classical Lie algs & some exceptional

Levasseur - Stafford JAMS 1996

- prove using trick from noncommutative algebra:

Proposition If a finite group Γ acts on vector space V $\Rightarrow$
$\mathcal{D}(V)^\Gamma$ is a simple algebra ("Maschke theorem"
- analog of $\mathbb{C}[\Gamma]$ semisimple)

Lemma (L-S): Let $R \subseteq S$ be two Noetherian rings (left & right)
without zerodivisors (not nec. commutative)
s.t.
- skew fields of fractions $Q(R) = Q(S)$ (Ore: $Q(R)$ exists)
- S is finite as both left & right R-module
- S is simple

$\Rightarrow$ $R = S$.

~~$\Rightarrow$ apply to $R = \mathrm{Im}(\phi)$.~~

$$\phi : \mathcal{D}(\mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h})^W$$

$$\mathrm{gr}\,\phi : \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}]^W$$

suffices to prove $\mathrm{gr}\,\phi$ surjective to show ϕ surjective.

Thm (A. Joseph '97) $\mathrm{gr}\,\phi$ surjective

Pf involves crystal bases for quantum groups.
- in fact can put any number of repres..

$$\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}]^W \supseteq \mathbb{C}[\mathfrak{h}_1]^W, \quad \supseteq \mathbb{C}[\mathfrak{h}_2]^W$$

both contained in image of maps... how much can we build from these?

- do they Poisson generate? - in most cases but not all exceptional ones.

V. Ginzburg 11/6/06

$L_k = \Delta - k(k+1) \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}$ diff op on $\mathbb{C}^n \setminus \text{diag} = \mathfrak{h}^{reg}$

$\mathcal{C}_k$ = centralizer of L_k in $\mathcal{D}(\mathfrak{h}^{reg})^W$

$\mathcal{B}_k$ = associative subalg of $\mathcal{D}(\mathfrak{h}^{reg})^W$ gen. by $\mathcal{C}_k$ & $\mathbb{C}[\mathfrak{h}]^W$

$\Phi_k : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h}^{reg})^W$

Theorem: $\text{Image}(\Phi_k) = \mathcal{B}_k$.

Don't know $\mathcal{C}_k$ explicitly - what kind of singularities?
know almost nothing about $\mathcal{B}_k$ - see complicated algebra...

- G acts on $\mathfrak{g}$ adjointly, so $\mathfrak{g} \to \text{Vect}\,\mathfrak{g}$, $x \mapsto ad\,x$ linear vector fields
  - extends to homomorphism $ad : U\mathfrak{g} \to \mathcal{D}(\mathfrak{g})$
- Rep $V_k = (x_1 \dots x_n)^k\, \mathbb{C}_{deg\,0}[x_1^{\pm 1}, \dots x_n^{\pm 1}]$ $\mathfrak{g}(=\mathfrak{sl}_n)$-module
  $\text{Ann}\, V_k \subset U\mathfrak{g}$ annihilator of V_k (two sided ideal in $U\mathfrak{g}$).
- $ad(\text{Ann}\, V_k) \subset \mathcal{D}(\mathfrak{g})$ subalgebra
- Let $I_k = \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \cap \mathcal{D}(\mathfrak{g}) \cdot ad(\text{Ann}\, V_k)$

Theorem For all but fin many [probably all] $k \in \mathbb{C}$, $\ker \Phi_k = I_k$.

Drinfeld Quantum analogue of Hamiltonian reduction:

M Poisson variety, $A_0 = \text{Fun}(M)$ $\{\,\}$ Lie bracket, derivation of commu. alg structure

Quantization: A algebra $/ \mathbb{C}[[h]]$, (flat &) topologically free
(in particular h-adically complete): $A = \varprojlim A/h^n A$ & $A/h^n A$ free $/ (\mathbb{C}[[h]]/h^n)$

$A/hA = A_0$, as Poisson algebras
(A/hA naturally Poisson)

$a, b \in A_0$ lift to $\tilde{a}, \tilde{b} \in A$ $\{a,b\} = \frac{[\tilde{a},\tilde{b}]}{h}$ mod h.

Classical: M Hamiltonian G-space, $G \times M \to M$, $\mu : M \to \mathfrak{g}^*$ Poisson map
st. 1. infi action $\mathfrak{g} \to \text{Vect}\, M$ agrees with composite
$\mathfrak{g} \hookrightarrow \text{Fun}(\mathfrak{g}^*) \longrightarrow \text{Fun}\, M \xrightarrow{\{\,\}} \text{Vect}\, M$
2. $M \to \mathfrak{g}^*$ is G-equivariant (automatic for G connected).

i.e. have Poisson algebra A_0, G action $G \times A_0 \to A_0$ &
Lie alg. homomorphism $\mathfrak{g} \to A_0$, with requirements
1. Infi action $\mathfrak{g} \times A_0 \to A_0$ is adjoint action wrt $\mathfrak{g} \to A_0$
2. $\mathfrak{g} \to A_0$ is G-equivariant.

Reduction: $\mathbb{O} \subset \mathfrak{g}^*$, $M /\!\!/_{\mathbb{O}} G = \mu^{-1}(\mathbb{O})/G$

Algebraic setting - rather work with orbit closure rather than just orbits, or in general any G-invariant subvariety of $\mathfrak{g}^*$.

Algebraic analog $\overline{\mathcal{O}} \rightsquigarrow I \subset \mathrm{Fun}(\mathfrak{g})$ ideal which is G-invariant $\iff$ I is a Poisson ideal (in general need G-invariant when G not connected) (i.e. have Poisson structure on quotient).

(over $\mathcal{O}$) Reduction: $A_0 \rightsquigarrow (A_0/IA_0)^G$ for general G... or can ~~take~~ take $A_0^G/(IA_0 \cap A_0^G)$: first always injects into latter, & for G reductive is surjective: need to extend function from subvariety to total space in G-invariant fashion, & can "average" for G reductive

[in quantum case harder to define nonaffine version $\rightsquigarrow \mathcal{O}$]

$$A_0^G/(IA_0 \cap A_0^G) \hookrightarrow (A_0/IA_0)^G, \text{ isom for } G \text{ reductive}$$

... $IA_0 \cap A_0^G$ is a Poisson ideal of A_0^G.

Third (& best) definition: homological version

$A_0 \otimes_{\mathrm{Fun}\,\mathfrak{g}^*} \mathrm{Fun}\,\mathfrak{g}^*/I \rightsquigarrow$ derived version, BRST

Quantum version: "quantize definitions", less precise than quantizing objects...

Def A assoc algebra, $G \times A \to A$, $\mathfrak{g} \to A$ with the two compatibilities. $I \subset U\mathfrak{g}$ two-sided ideal (analog of Poisson ideal), G-invariant $\rightsquigarrow A^G/(IA \cap A^G)$ why is intersection two-sided ideal?

Assume G reductive, or in particular A is a semisimple G-Module.

Lemma $IA \cap A^G = AI \cap A^G$, $IA \cap A^{\mathfrak{g}} = AI \cap A^{\mathfrak{g}}$

Pf Average operator: $\pi: A \longrightarrow A^{\mathfrak{g}}$

$IA \cap A^{\mathfrak{g}} = \pi(IA)$

- $g \in \mathfrak{g}$: $\pi(ga) = \pi(ag)$ $\quad (\pi([g,a]) = 0 \quad A = A^{\mathfrak{g}} + \mathrm{Im}(\mathrm{ad}\,\mathfrak{g}))$
- $\pi(g_1 g_2 a) = \pi(g_2 a g_1) = \pi(a g_1 g_2)$

so $\pi(ua) = \pi(au) \quad \forall u \in U\mathfrak{g} \quad (U\mathfrak{g} \to A)$.

$\Longrightarrow \pi(IA) = \pi(AI)$ ∎

Ginzburg

Suppose X is a G-manifold. $\rightsquigarrow T^*X$ is Hamiltonian G-space

$\mu: T^*X \longrightarrow \mathfrak{g}^*$ adjoint of action map $\mathfrak{g} \to TX$.

Quantization of $\mathbb{C}[\mathfrak{g}^*]$ is $U\mathfrak{g}$, $\mathbb{C}[T^*X]$ is $\mathcal{D}(X)$

- filtered assoc algebras with gr commutative $\rightsquigarrow$ gr is Poisson.

Want $U: U\mathfrak{g} \to \mathcal{D}(X)$, quantization of μ^*

- comes from action $\mathfrak{g} \to \mathrm{Vect}\, X$.

Reduction : $A^G / (A^G \cap A\,\nu(I))$

Special case : $X = \mathfrak{g}$, G acts by adjoint action

$\nu = \mathrm{ad} : U\mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{g})$.

$\mathcal{D}(\mathfrak{g})^{\mathfrak{g}} / (\mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \cap \mathcal{D}(\mathfrak{g}) \cdot \mathrm{ad}\, I)$ -- Thm : $\mathrm{Ker}\, \phi_k = \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \cap \mathcal{D}(\mathfrak{g}) \mathrm{ad}(\mathrm{Ann}\, V_k)$

Take $I = \mathrm{Ann}\, V_k$: $U\mathfrak{g} / \mathrm{Ann}\, V_k$ is quantization of Fun (conj class of $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & -n \end{pmatrix}$)

- orbit itself is twisted cotangent bundle.

TDOs quantization of twisted cotangent bundle, $U\mathfrak{g}$ maps to global twisted differential operators ...

I quantum analog of ideal of functions on $\mathfrak{g}^*$: I is kernel of map from $U\mathfrak{g}$ to twisted diffops (level k) on $\mathbb{P}^{n-1}$!

"Classical" case $k=0$: $\phi_0 = \phi$ Classical Harish-Chandra

$(\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}) \longrightarrow \mathcal{D}(\mathfrak{h})^W$ What is kernel?

Solved by Levasseur-Stafford ! (conjectured by Dixmier)

<u>Theorem</u> $\mathrm{Ker}\, \phi = \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \cap \mathcal{D}(\mathfrak{g})\, \mathrm{ad}\, \mathfrak{g}$

(= Hamiltonian reduction over 0)

-- what one would expect from radial part construction: kernel should be things which annihilate invariant functions.

Highly nontrivial !

<u>Commutative analogues</u> : $\mathrm{gr}\,(\mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \cap \mathcal{D}(\mathfrak{g})\, \mathrm{ad}\, \mathfrak{g})$

$= (\mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]^G \cap \mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]\ \mathrm{ad}\ \mathfrak{g})$

Zero variety of $\mathbb{C}[\mathfrak{g} \oplus \mathfrak{g}]\, \mathrm{ad}\, \mathfrak{g}$:

moment $\mu : \mathfrak{g} \oplus \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is $x, y \mapsto [x, y]$.

"ad $\mathfrak{g}$" : matrix elements of commutators, as fns on $\mathfrak{g} \oplus \mathfrak{g}$

- i.e. $\{ \lambda([x,y]),\ \lambda \in \mathfrak{g}^*,\ x, y \in \mathfrak{g} \}$

- i.e. equations for $[x,y] = 0$.

So zero variety is $\{ (x,y) \in \mathfrak{g} \oplus \mathfrak{g} \mid [x,y] = 0 \} = Z$, the commuting variety.

$\mathrm{Ann}\, V_k \subset U\mathfrak{g} \rightsquigarrow \mathrm{gr}\, \mathrm{Ann}\, V_k \subset \mathrm{gr}\, U\mathfrak{g} = \mathbb{C}[\mathfrak{g}]$.

- this is the ideal (quadratic) defining cone

$x_{ij} x_{kl} - x_{ik} x_{jl}$ - matrices of rank ≤ 1.

Zero variety of $\mathrm{gr}(\mathrm{Ann}\, V_k)$ is nilpotent matrices of rank ≤ 1

- rescale Calogero-Moser variety & take parameter to zero

$\rightsquigarrow$ asymptotic cone of $\mathcal{O}_{CM}$.

Zero variety of $\mathbb{C}[\mathfrak{g}\oplus\mathfrak{g}] \cdot \mathrm{gr}(\mathrm{ad}\,\mathrm{Ann}\, V_k) =$

$\mathcal{Z}_1 = \{(x,y) \in \mathfrak{g}\oplus\mathfrak{g} \mid [x,y] = \text{nilpotent of rank} \leq 1\}$

- obvious discrepancy with $\mathcal{Z}$! $\quad \mathcal{Z}_1 \supset \mathcal{Z}$

~ i.e. Levasseur-Stafford not graded ?

$\boxed{\text{Open problem}}$: is the ideal $J = \mathbb{C}[\mathfrak{g}\oplus\mathfrak{g}]^{\mathfrak{g}} \cap \mathbb{C}[\mathfrak{g}\oplus\mathfrak{g}]\,\mathrm{ad}\,\mathfrak{g}$ radical ? $\quad \sqrt{J} \overset{?}{=} I$

i.e. $\mathfrak{g}$ ideal in $\mathbb{C}[\mathfrak{g}+\mathfrak{g}]$ given by eqs $[x,y]=0$ prime ? (known that variety is irreducible).

.. not complete intersection..

This would imply theorem immediately.

~~Ann solution~~

Levasseur-Stafford proof $\quad$ Let $\cancel{I} = \cancel{\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}} \cap (\mathcal{D}(\mathfrak{g})\,\mathrm{ad}\,\mathfrak{g}) = J$ just left ideal.

Consider $M = \mathcal{D}(\mathfrak{g}) / \mathcal{D}(\mathfrak{g})\,\mathrm{ad}\,\mathfrak{g}$, as $\mathcal{D}$-module on $\mathfrak{g}$.

- singular support $SS(M) = \mathcal{Z} \subset T^*\mathfrak{g}$.

$\dim \mathcal{Z} = \dim \mathfrak{g} + \mathrm{rk}\, \mathfrak{g}$ $\quad$ (typical pairs: two simultaneously diag. matrices) $\quad$ - not holonomic!

The algebra $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$ commutes with $\mathrm{ad}\,\mathfrak{g}$, therefore acts on M on right.

Easy to see by construction (tautologically) that the right $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$ action descends through $\mathrm{Ker}\,\phi$, hence to $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}/\mathrm{Ker}\,\phi \simeq \mathrm{Im}\,\phi = \mathcal{D}(\mathfrak{h})^W$

$\rightsquigarrow$ classical analog is $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}]^W$, which is singular

Thus M is a $\mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{h})^W$-module.

If we didn't have W-invariant ~ if we had $\mathcal{D}[\mathfrak{g}\times\mathfrak{h}]$

M would be holonomic over this.. $\mathcal{D}(\mathfrak{g})\otimes\mathcal{D}(\mathfrak{h})^W$ is not diff ops on anything, but M behaves as if it were holonomic ...

Obvious that $\quad \mathrm{Ker}\,\phi \supset \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \cap \mathcal{D}(\mathfrak{g})\,\mathrm{ad}\,\mathfrak{g}$

- not equality: if Ker was bigger you'd construct a smaller $M \to M'$, & holonomicity & mult 1

$\Rightarrow$ no smaller M''!

— use nice criterion for irreducibility of $\mathcal{D}$-mod!

only need mult. 1 (no nilpotents) at the generic point in some sense unlike classically where we need no nilpotents (radical ideal) everywhere.

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irred $\quad Z = \{ A, B \in \mathfrak{gl}_n \times \mathfrak{gl}_n \mid [A,B]=0 \}$
$\qquad\quad\ \wedge$
many red. components $\quad Z_1 = \{ A, B \in \mathfrak{gl}_n \times \mathfrak{gl}_n \mid [A,B] \text{ nilpotent rank one} \}$

$\mathbb{C}[Z_1] \to \mathbb{C}[Z]$

$\boxed{\text{Theorem}}$ $\quad \mathbb{C}[Z_1]^{GL_n} \xrightarrow{\sim} \mathbb{C}[Z]^{GL_n}$ — resolves contradiction from last time. (fns as reduced varieties)

"Barth's lemma" Lemma Let A, B be $n \times n$ matrices s.t. $[A,B]=C$, rk $C=1$, C nilpotent.
$\Rightarrow A, B$ can be simultaneously upper-triangularized
(i.e. A, B generate solvable Lie algebra)

Tyurin solution: $\quad A \longrightarrow A - \lambda \, Id$ if necessary $\to$ assume
$\operatorname{Ker} A \neq 0, \quad \operatorname{Im} A \subsetneq \mathbb{C}^n$
By induction on n, suffices to show $\exists \, W \subset \mathbb{C}^n$ proper A, B-stable. $\qquad$ Exercise: W is either $\operatorname{Ker} A$ or $\operatorname{Im} A$.
– one of these is B-stable ... ∎

Remark $\quad$ Given $\mathfrak{g}$ semisimple Lie algebra, $A, B \in \mathfrak{g}$ s.t.
$[A,B]$ is a root vector e_α (GL_n: all these are rank one)
— does $\boxed{\text{not}}$ always imply A, B generate a solvable Lie algebra. $\quad$ – counterexample in $so(8)$

By Weyl, all GL_n invariant polynomials on $\mathfrak{gl}_n \times \mathfrak{gl}_n$ are generated by functions like
$(A,B) \longmapsto tr(A^{k_1} B^{l_1} A^{k_2} \cdots)$ $\quad$ as an algebra.

Z_1, Z closed subvarieties – G-inv't fns on these extend to invariant fns on $\mathfrak{gl}_n \times \mathfrak{gl}_n$ (GL_n reductive!)
so if we upper triangularize, get same functions as if they commute: only diagonal entries come into $tr(A^{k_1} B^{l_1} \cdots)$. ∎