V. Ginzburg 11/16/00

Symplectic Reflection Algebra

V vector space / $\mathbb{C}$ (f.d.)

$\Gamma \subset GL(V)$ finite subgroup.

Take tensor algebra TV over V, consider smash

$TV \# \Gamma \underset{\text{vect space}}{=} TV \otimes_{\mathbb{C}} \mathbb{C}\Gamma$,

& ~~$g\cdot v = v\cdot g \cdot g(v)$~~ $g\cdot v \cdot g^{-1} = g(v)$ for $v\in V, g\in\Gamma$.

Assume given a skew-symmetric bilinear form $\beta : V\otimes V \longrightarrow \mathbb{C}\Gamma$.

$\Rightarrow$ take $TV \# \Gamma / \langle v_1\otimes v_2 - v_2 \otimes v_1 - \beta(v_1 \otimes v_2)\rangle =: H_\beta$

If $\beta = 0$: $H_0 \simeq SV \# \Gamma$ SV symmetric algebra.

$\rightarrow$ a graded algebra, $(SV\#\Gamma)_0 = \mathbb{C}\Gamma$, $(SV\#\Gamma)_1 = V\otimes\mathbb{C}\Gamma$, ...

$(SV\#\Gamma)_k = S^k V \otimes \mathbb{C}\Gamma$ algebra grading.

H_β in general is filtered, with some deg 0 component is image of $\mathbb{C}\Gamma$ - could be smaller.

Have well-defined homomorphism $\varphi : SV\#\Gamma \twoheadrightarrow gr\, H_\beta$

Definition Say that H_β satisfies PBW property if φ is an isomorphism. ("reasonable")

Assume (V,ω) symplectic vector space, $\Gamma\subset Sp(V)$ finite subgroup.

Say that a triple (V,ω,Γ) is ~~indecomposable~~ if there is no Γ-stable nontrivial orthogonal decomposition $V = V_1 \oplus V_2$, V_i symplectic.

2 alternatives:

- V is a irreducible Γ-module of quaternionic type (with symplectic form)
- $V = h\oplus h^*$ where h is a simple Γ-module of either real or complex type

quaternionic ~ symplectic group (skew symmetric)

real - orthogonal group (symmetric form) complex ~ GL_n (no form)

In all such cases, $\dim (\Lambda^2 V)^\Gamma = 1$.

Definition An element $s\in\Gamma$ is called a symplectic reflection if $rk(Id - s) = 2$

Γ finite so any such s is diagonalizable

$\Rightarrow s = \begin{pmatrix} \lambda & & & \\ & \lambda^{-1} & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$ $\lambda\neq 1$

Complex reflection: $\mathrm{rk}(Id - s) = 1$ — doesn't occur in symplectic situation — for finite subgroups of $GL(V)$.

Let S = set of symplectic reflections in Γ.

Γ acts on S by conjugation.

Theorem Assume (V, ω, Γ) indecomposable triple, & $\beta : V \wedge V \to \mathbb{C}\Gamma$.

Then PBW property holds for $H_\beta \iff$

$\exists\, t \in \mathbb{C}$ & an $\mathrm{Ad}\,\Gamma$-invariant function $c : S \to \mathbb{C}$, $s \mapsto c_s$

s.t. $\beta(v_1, v_2) = t\,\omega(v_1, v_2)\cdot 1 + \sum_{s \in S} c_s\, \omega_s(v_1, v_2) \cdot s \in \mathbb{C}\Gamma$

- in particular only symplectic reflections occur.. Also β automatically Γ-invt.!

If $s \in S \Rightarrow V = \underbrace{\mathrm{Im}(Id - s)}_{2\text{-dim}} \oplus \mathrm{Ker}(Id - s)$

ω_s is a skew form on V defined by $\omega_s|_{\mathrm{Im}(Id-s)} = \omega$

& zero on $\mathrm{Ker}(Id-s)$ $(\mathrm{Rad}(\omega_s) = \mathrm{Ker}(Id - s))$ -- just form on two-dim space.

Examples: • $c = 0 \Rightarrow$ Weyl algebra $A_t = TV / \langle v_1 \otimes v_2 - v_2 \otimes v_1 - t\omega(v_1, v_2)\rangle$

For $t \neq 0$, $\simeq \mathcal{D}(\mathbb{A}^n)$.

$H_\beta = H_{t,c} \simeq A_t \# \Gamma$ in this case.

-- so introducing c gives a deformation of $A_t \# \Gamma$.

(by PBW size of algebra doesn't change

(well understood) •1. Kleinian singularities: $V = \mathbb{C}^2$, $\Gamma \subset SL_2\mathbb{C}$ finite

Every $1 \neq g \in \Gamma$ automatically a symplectic reflection

finite subgroups in SL_2 same as in SU_2 — mod center

get SO_3 $\quad A_n$ cyclic $\quad D_n$ dihedral $\quad E_6\ E_7\ E_8 \to$ five platonic solids.

$\omega = \omega_s$.

$\beta(x, y) = \omega(x, y) \cdot Z$,

$Z \in \mathrm{Center}(\mathbb{C}\Gamma)$ arbitrary.

x, y symplectic basis of $\mathbb{C}^2$.

$H_\beta = TV \# \Gamma / xy - yx = Z$. - replace scalar by central element in $\mathbb{C}\Gamma$

- i.e. ~~$\mathcal{U}(\mathrm{Heis}_3) \otimes \mathbb{C}\Gamma$~~

(unexplored) 2. $V = \underbrace{(\mathbb{C}^2) \oplus \dots \oplus (\mathbb{C}^2)}_{n}$, Fix $\Gamma' \subset SL_2\mathbb{C}$ Kleinian

$\Gamma = S_n \ltimes (\underbrace{\Gamma' \times \dots \times \Gamma'}_{n})$

- always generated by symplectic reflections (since S_n gen. by transpositions).

3. $\Gamma = W$ finite Coxeter group e.g. Weyl groups of an irred root system $R \subset \mathfrak{h}^*$. $V = \mathfrak{h} \oplus \mathfrak{h}^*$, Γ acting diagonally.
Symplectic reflection in $V \Longleftrightarrow$ ordinary reflection in $\mathfrak{h}$
so $s_\alpha \in S \longleftrightarrow$ set of roots α

At most two W-orbits on set of roots (given by lengths)
so c is either 1 or 2 numbers.
$H_{t,c}$ is generated by $y \in \mathfrak{h}$, $x \in \mathfrak{h}^*$, $w \in W$ with relations
$wxw^{-1} = w(x)$, $wyw^{-1} = w(y)$, $[x_1, x_2] = 0$, $[y_1, y_2] = 0$

& $yx - xy = t \cdot \langle x, y \rangle \cdot 1 + \sum_{\alpha \in R} c_\alpha \langle y, \alpha \rangle \langle \alpha^\vee, x \rangle \cdot s_\alpha$

Attempt to generalize deformed Weyl algebra from two dim to others as $[x, y] = Z \in$ Center $(\mathbb{C}\Gamma)$ doesn't work in dim > 2!

Def Idempotent $e = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g$ standard averaging operator
Call $eH_{t,c}e$ the spherical subalgebra in $H_{t,c}$.

Theorem 1 The spherical subalgebra is commutative iff $t = 0$.
Theorem 2 (Satake isomorphism) Center $Z(H_{0,c}) \xrightarrow{\sim} eH_{0,c}e$, $z \mapsto ze$

Motivation: simplest case $H_{0,0} = (SV) \# \Gamma$,
$Z(H_{0,0}) = (SV)^\Gamma = eH_{0,0}e = \mathbb{C}[V/\Gamma]$

Corollary Spec $Z_{0,c}$ form a deformation of V/Γ depending on $n(1) = (\#\ \Gamma$-conj. classes of in $S)$ parameters.
(universal in Kleinian case - seems to be only case where singularity is isolated. in other cases deforms away all singularity).

For each k let $n(k) = \#\ \Gamma$ conj classes in the set $\{g \in \Gamma \,/\, rk(Id - g) = 2k\}$
... $n(1) = \#(S/\Gamma)$.

J. Alev et al
J. of Algebra
2000

Theorem V symplectic vector space, $\Gamma \subset Sp(V)$ finite, Hochschild cohomology
$\dim HH^i(A_t^\Gamma) = \begin{cases} 0 & i \text{ odd} \\ n(k) & i = 2k \end{cases}$
(Computation!)

Corollary $\dim HH^2(A_t^\Gamma) = n(1)$ = dim of tangent space to deformation space of A_t^Γ in associative algebras:
$HH^2(B) = T_B$ ($\mathcal{M}$ = moduli of assoc algebras).

$eH_{t,c}e$ ~~$H_{t,c}$~~ produces family of right dimension!

Theorem For any given $t \neq 0$ the spherical family $\{eH_{t,c}e\}_c$ is a universal deformation of A_t^Γ.

Kontsevich: $HH^{even}(B)$ = tangent at B to moduli of A_∞-algebras.

Question: Construct natural family of A_∞ structures on $eH_{t,c}e$. --- all odd part vanishing means binary operation remains associative in deformation

Remark For any c the algebra $eH_{0,c}e$ is Poisson, coming from deformations $eH_{t,c}e$.

Conjecture The family $eH_{0,c}e$ is a universal deformation of the Poisson algebra $(SV)^\Gamma$ in Poisson category

Mirror symmetry: look for ~~smooth~~ crepant resolutions $\widehat{V/\Gamma} \to V/\Gamma$

Theorem (Batyrev): If $\widehat{V/\Gamma}$ is a crepant resolution of V/Γ $\Rightarrow$ Betti numbers $\dim H^{2i}(\widehat{V/\Gamma}) = n(i)$ (using motivic integration).

$$\widehat{V/\Gamma}$$
$$\downarrow$$
V/Γ $\rightsquigarrow$ deformation space $eH_{0,c}e$: expect to lift to deformation of $\widehat{V/\Gamma}$ & Betti numbers to be same as for $\widetilde{V/\Gamma}$.

Known in Kleinian & Calogero-Moser case.

Hope: For generic c, $X = \operatorname{Spec} eH_{0,c}e$ is smooth & symplectic (Poisson structure nondegen).

$H^\bullet(X) \simeq$ Poisson cohomology $PH^\bullet(\mathbb{C}[X])$ (Brylinski)

degeneration ~~deformation~~ of Hochschild cohomology
– tangent space to symplectic space $\Longleftrightarrow H^2(X)$ $\Longleftrightarrow HH^2$.

& expect $PH^\bullet(\mathbb{C}[X]) \overset{?}{\simeq} HH(A_t^\Gamma)$

$\hbar \rightsquigarrow$ noncommutative geometry deformation in physics...

V. Ginzburg - Symplectic Reflection Algebras II 11/20/00

Recall $\gamma\beta(x,y) = t\omega(x,y) + \sum_{s\in S} c_s \omega_s(x,y)\cdot s \in \mathbb{C}\Gamma$,

Algebra $H_{t,c} = TV \# \Gamma / x\otimes y - y\otimes x - \gamma\beta(x,y)$

S = set of symplectic reflections, $c: S \to \mathbb{C}$ Γ invariant.

Remark: $e = \frac{1}{|\Gamma|}\sum_{\Gamma} g$ eHe spherical subalgebra

~~Theorem~~ Situation parallel to affine Hecke case:

H = affine Hecke algebra, $\Gamma = W$, eHe = spherical subalgebra

$\cup H_{fin}$ deformation of $\mathbb{C}\Gamma$ (fin dim algebra) so can assume

$H \supset \mathbb{C}W$, e = canonical W-idempotent. $eHe = \mathrm{Hecke}(G(K), G(G))$

spherical Hecke algebra. $Z(H) \xrightarrow{\sim} eHe$. (Satake)

We have same picture when $t=0$.

Consider $H_{t,c}e$ left H-module, right eHe module. For any algebra

~~Theorem~~ A & right A-module M, set ${}^{\vee}M = \mathrm{Hom}_{A\text{ right}}(M,A)$ ~~right~~ left

$M^{\vee} = \mathrm{Hom}_{left}(M,A)$ for left is right

Theorem (i) $eH_{t,c}e$ is a finitely generated Gorenstein algebra (without zero-divisors)

(ii) $H_{t,c}e$ is f. gen Cohen-Macaulay $eH_{t,c}e$-module

(iii) ${}^{\vee}(H_{t,c}e) \simeq eH_{t,c}$ & $(eH_{t,c})^{\vee} = H_{t,c}e$

as spherical modules.

In particular $H_{t,c}e$ is reflexive

(iv) $H_{t,c} \longrightarrow \mathrm{End}_{eH_te}(H_te)$ (from left action) is an isomorphism

---

$\therefore$ will mainly use for $t=0$, algebras commutative $\Rightarrow$ usual notions of Gorenstein, Cohen-Mac.

For any Noetherian (noncomm) algebra A $\Rightarrow$ introduce notion of dualizing complex $D_A \in D^b(A\text{-bimodules})$.

- not enough morphisms to determine completely from an $f^!$ condition...

D^0 - want each cohomology group fin gen as both left & right A-mod.

Want quism $A \xrightarrow{\sim} R\mathrm{Hom}_{A\text{-}A}(D_A, D_A)$

- doesn't determine D_A (even up to shift & line bundle as in commutative case).

Van den Bergh: puts a non-linear condition determining D_A uniquely if it exists. — rigid dualizing complex.

For filtered algebras with graded commutative — rigid D_A exists (lift from graded) (see also Yekutieli, Zhang)

Def: M Cohen-Macaulay of deg d if $\mathrm{Ext}^i_{\text{left } A}(M, D_A) = 0 \ \forall i \neq d$

A ~~Gorenstein~~ Cohen-Macaulay if A C-M as A-module (left)

i.e. $H^i(D_A) = 0 \ \forall i \neq 0$

A Gorenstein if A C-M & $A \simeq H^0(D_A)$

... A quism to D_A.

<u>Proof</u> of PBW $\Longleftrightarrow$ β has above form:

$A = (TV) \rtimes \Gamma$ is a graded algebra, $A_0 = \mathbb{C}\Gamma$

Let $E = V \underset{\mathbb{C}}{\otimes} \mathbb{C}\Gamma$, treat as A_0-bimodule.

right action obvious, left action simultaneous:

$g(v \otimes g_1) = gv \otimes g g_1$, $(v \otimes g_1) g = v \otimes g_1 g$

$-$ E A_1 graded piece

Then $TV \rtimes \Gamma = T_{A_0} E$ tensor algebra over A_0.

$\beta : V \otimes V \to \mathbb{C}\Gamma$ can be extended to an A_0-bilinear form

$\beta : E \underset{A_0}{\otimes} E \to A_0 = \mathbb{C}\Gamma$.

$$H_\beta = \frac{TV \rtimes \Gamma}{x \otimes y - y \otimes x - \beta(x,y)} = \frac{T_{A_0} E}{\text{relation in } T^2_{A_0} E + A_0}$$

Thm (Drinfeld ~86 (ref in Braverman-Gaitsgory J. Alg '96 on PBW))

Selecta old series '92

If $T_{A_0} E$ / (homog quadratic relation) is Koszul

$\Longrightarrow$ (under certain explicit conditions) PBW holds for $T_{A_0} E$ / non-homog relations (homog + lower).

(Drinfeld) Interested in following: V f.d. vector space

$\mathrm{Sym}\, V = TV / (\text{ideal}_{\text{2-sided}} \text{ gen by } \Lambda^2 V \subset V \otimes V)$

Suppose $L_t \subset V \otimes V$ family of subspaces, $L_0 = \Lambda^2 V$

(t "small" - over local ring)

$A_t = T(V) / \text{ideal gen by } L_t$ family of graded algebras,

$A_0 = \mathrm{Sym}\, V$.

When is this a deformation - i.e. flat in t -

i.e. $\dim A_t^n$ indep of t for t suff. small

(depending on n)

PBW: $\dim A_t^n = \dim A_0^n$ small t.

<u>Theorem</u> If equality holds for $n=3$ & small $t \Rightarrow$ holds for all n.

(can reduce nonhomogeneous case to this ...)

& can give explicit equations on L_t for this to hold.

(Ginzburg) In our case working over A_0, noncom — important point is that it's a semisimple algebra.

Consider $\beta \otimes Id - Id \otimes \beta : (E \wedge E) \otimes_{A_0} E \cap E \otimes (E \wedge E) \to E$

- its vanishing is Braverman-Gaitsgory condition for vanishing.

Our case: $E \wedge E \otimes E \cap E \otimes E \wedge E = \Lambda^3 V \otimes \mathbb{C}\Gamma \subset V^{\otimes 3} \otimes \mathbb{C}\Gamma$

(skew in first two args & skew in second two).

$x, y, z \in V$ this vanishing means

$Alt([xy]z - x[yz]) = 0$ $\qquad [xy] = xy - yx$

--- precisely Jacobi identity $\qquad [z[xy]] = [[zx]y] + [x[zy]]$

- Want Jacobi to hold $\rightsquigarrow$ conditions on β

$$[x,y] = \beta(x,y) = \sum_{g \in \Gamma} b(x,y,g) \cdot g \in \mathbb{C}\Gamma$$

$$\sum_g b(x,y,g)(z - z^g) \cdot g = \sum_g b(z,x,g)(y - y^g) g + \sum b(z,y,g)(x - x^g)$$

$z^g = g \cdot v$ $\quad \Gamma$ action on V — want this for all g

$$b(x,y,g)(Id - g)(z) = b(z,x,g)(Id - g)y + b(z,y,g)(Id - g)x.$$

Fix $g \in \Gamma \Rightarrow x, y \mapsto b(x,y,g)$ ~~nonze~~ bilinear scalar form on V

Suppose nonzero form $\Rightarrow$ nonzero for some x, y

$\Rightarrow$ image $(Id - g)z$ is the span of ~~two vecto~~ generic $y^g - y$ & $x^g - x$: 2-dimensional!

$\Rightarrow$ symplectic reflection

Reminder on Poisson brackets

Suppose (V, ω, Γ) indecomp. triple, $\dim(\Lambda^2 V)^\Gamma = 1$.

Lemma Consider homogeneous Poisson bracket B on $(SV)^\Gamma$ of degree l: $\quad B : (SV)^\Gamma_i \otimes (SV)^\Gamma_j \mapsto (SV)^\Gamma_{i+j+l}$

Then $B = t \cdot \{\}$ standard Poisson (from ω) if $l = -2$, & B vanishes if $l < -2$

$$\{\} = \frac{\partial P}{\partial x}\frac{\partial Q}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial Q}{\partial x} \qquad \deg = -2.$$

Pf B is given by a bivector, at least on smooth locus $(V/\Gamma)^{res}$.
$V^{reg} \longrightarrow (V/\Gamma)^{reg}$. $\quad V \setminus V^{res}$ = set where Γ action not free
$= \bigcup_{g \in \Gamma} V^g$ $\quad V^g$ always hyperplane of codim ≥ 2 ($\Gamma < SL(V)$: can't have just one e'val $\neq 1$)
So bivector B extends everywhere, $\Rightarrow$ Γ-inv't bivector on V. So if deg -2 $\Rightarrow$ Γ inv't elem't of Λ^2.
– Can't have B homog of negative degree on SV & defined everywhere ...

Suppose A_t flat deformation of a comm algebra A_0.
$\rightsquigarrow$ $\mathcal{A}$ flat / $\mathbb{C}[[t]]$ ...
$a, b \in A_0 \rightsquigarrow$ lift to $\tilde{a}, \tilde{b}$, $\Rightarrow \{a,b\} = \dfrac{\tilde{a}\tilde{b} - \tilde{b}\tilde{a}}{t}$ mod tA.

May happen that $\{a,b\}$ identically zero...
more generally if $[\tilde{a}, \tilde{b}] \in t^k \mathcal{A}$ all $\tilde{a}\ \tilde{b}$,
take t^k minimal possible
$\Rightarrow \{a,b\} = \dfrac{[\tilde{a}, \tilde{b}]}{t^k}$ cannot vanish (unless $\mathcal{A}$ commutative)

Consider $eH_{t,c}e$, write with h parameter $xy - yx = \beta(x,y)\,h$,
deg $h = 2$. Fix c, t & treat h as deformation parameter.

$h = 0 \Rightarrow eH_{tc}e = e(SV \# \Gamma)e = SV^\Gamma$ commutative
The family $eH_{t,c}e$ gives a def of SV^Γ
$\Rightarrow$ a Poisson bracket $B_{t,c}$ on SV^Γ.
Degree of homogeneity depends on k_{tc} we need in def of bracket $\dfrac{[\ ]}{t^k}$.

By Lemma will be zero $\{\}$ unless $k_{tc} = 1$
– i.e. $eH_{tc}e$ noncommutative iff $k_{tc} = 1$

So $xy - yx = (t\,\omega(x,y) + \sum c_s\, \omega_s(x,y)\, s)\, h$ $\quad$ deg t = deg

$c \in \mathcal{C} = \mathbb{C}$-v.s. of Γ-inv't fns on S

<u>Claim</u> $B_{t,c} = f(t,c) \cdot \{\}_\omega$, f linear in c, t
(i.e. linear on $\mathbb{C} \times \mathcal{C}$)

- follows by homogeneity in t, c just as we used in t before!

- $eH_{t,c}e$ is commutative $\Longleftrightarrow f(t,c)=0$; some hyperplane D in our parameter space. (easy to show not always commutative)
Want to prove this is hyperplane $t=0$:
Claim $f = \text{const} \cdot t$. $\quad h \in D$

$F = H_{t,c}e$ is a (fin gen) $eH_{t,c}e$ module, along our hyperplane.
get sheaf on $\operatorname{Spec}(eH_{t,c}e)_h$ depends on point $h \in D$.

$\Gamma \subset H_{t,c}$, Γ acts fibrewise (commutes with eHe) on sheaf.
$\Rightarrow$ sheaf of Γ-modules.
SV as sheaf over V/Γ has generic fibre regular rep of Γ.
For any simple Γ-module χ, denote F_χ
the χ-isotypic component of F. $F_\chi = \operatorname{Hom}_\Gamma(\chi, F)$

- each gives sheaf on $\operatorname{Spec} eH_{t,c}e$.
Rank of F_χ on generic point of $\operatorname{Spec} eH_{t,c}e$
is <u>independent</u> of pt $h \in D$ — by PBW (flatness)

<u>Corollary</u>: generic fiber of F is the regular rep of Γ
(all multiplicities are same as for regular rep)
- we know it for $h=0$.

$[x,y] = t\omega(x,y) + \sum_s c_s \omega_s(x,y) s$. Acts fibrewise on our sheaf, in particular on generic fiber = regular rep.
Take trace on both sides:
on LHS $\operatorname{tr}[x,y]=0$
on RHS only identity has nonzero trace on regular rep
$\Rightarrow \operatorname{tr}(RHS) = t\,\omega(x,y) \Longrightarrow t=0$!

$\Rightarrow$ <u>Theorem</u> $eH_{t,c}e$ is commutative iff $t=0$. ∎
[Don't need our f to be linear — just $\operatorname{div} f \neq \emptyset$ that's enough — hypersurface contained in $t=0$]

$t=0$: by previously stated thm, $H_{0,c}e$ is a Cohen-Macaulay sheaf on $\operatorname{Spec} eH_{0,c}e$.

Hope: if c is generic enough then Spec $eH_{0,c}e$ smooth

$\Rightarrow H_{0,c}e$ is locally free $\dashrightarrow$ vector bundle $\mathcal{R}$ on Spec $eH_{0,c}e$
generically regular rep $\rightsquigarrow$ $\mathcal{R}$ carries regular rep of Γ
everywhere.
- don't know direct construction of $\mathcal{R}$ even in Kleinian case
(rk $\mathcal{R}$ = $|\Gamma|$) [Kleinian: $\mathfrak{C}$ = Cartan of ADE, smooth off reflection hyperplanes]

Theorem Suppose Spec $eH_{0,c}e$ smooth
$\Rightarrow$ for any point $x \in$ Spec eHe, the geometric fiber of $\mathcal{R}$ at x is a simple $H_{0,c}$ module
& assignment $x \mapsto \mathcal{R}_x$ gives bijection between
Spec $eH_{0,c}e$ & irreps of $H_{0,c}$.
(in particular all have dim $= |\Gamma|$.)

Proof We know $H \xrightarrow{\sim} \mathrm{End}_{eHe}(He)$ as stated (not yet proved).
$= \mathrm{End}\, \mathcal{R}$
So H is Morita equivalent to $\mathbb{C}[\mathrm{Spec}\, eHe]$
- endos of a vector bundle!

V. Ginzburg - Symplectic Reflection Algebras III 11/27/00

Geometric $^{(pre)}$ Quantization à la Kostant
(M, ω) symplectic, L line bundle on M, ∇ connection
$\mathrm{curv}(\nabla) = \omega$

$f \in \mathcal{O}(M) \longmapsto \hat{f} = f + \nabla_{\xi_f}$ operator on L

$\xi_f = \{f, \_\}$

$\Rightarrow$ Lie algebra homomorphism $(\mathcal{O}(M), \{\}) \to \mathrm{End}\, L$.

[Seek matrix version of this] $\wedge$ Satisfies $\hat{1} = \mathrm{Id}$: this requirement fixes things. $\hat{f} = f + \hbar \nabla_{\xi_f}$ will work for $\hbar\{\}$

$\Rightarrow$ get classical limit.

$H_{t,c}$ symplectic reflection algebra, $eH_{0,c}e$ commutative.

If $M = \mathrm{Spec}\, eH_{0,c}e$ is smooth $\Rightarrow$ $H_{0,c}e$ gives a rank $|\Gamma|$ vector bundle $\mathcal{R}_c$, $H_{0,c} \simeq \mathrm{End}\,\mathcal{R}$

Conj (Assuming M smooth) $\exists$ one-param. family of assoc algebra homomorphisms

$K_t : H_{t,c} \to \mathcal{D}iff(\mathcal{R}_c)$ s.t.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ id\ } & \Gamma \subset \mathrm{End}_{\mathcal{O}}\mathcal{R} \\ TV \supset V & \longrightarrow & \mathcal{D}_{\leq 1}(\mathcal{R}) \end{array}$$

& K_0 is the map $H_{0,c} \xrightarrow{\sim} \mathrm{End}\,\mathcal{R}$.

Q: When you deform an endo algebra can you deform it within diffops? expect so. What is analog of Poisson bracket in deformations of End algebras rather than comm. alg? (pictures ~~s~~ are Morita equivalent...)
"noncom. quantization" picture

From now on $V = \mathfrak{h} \oplus \mathfrak{h}^*$, $\Gamma = W$ Weyl group of finite root system $R \subset \mathfrak{h}^*$. V splits into two as Γ module.
Kleinian case $V = \mathbb{C}^2$: have such splitting only for cyclic group. So W is complex reflection group whenever have splittings, so might as well assume it Weyl group.

PBW thm: $H_{t,c} \overset{\text{vect space}}{=} S\mathfrak{h}^* \otimes_{\mathbb{C}} S\mathfrak{h} \otimes \mathbb{C}W$ all three subalgebras

$x \in \mathfrak{h}^*, y \in \mathfrak{h}$ $\quad yx - xy = t\langle x, y\rangle - \frac{1}{2}\sum_{\alpha \in R} c_\alpha \langle x, \alpha^\vee\rangle\langle \alpha, y\rangle s_\alpha$

Ex sl_2 $W=\{1,s\}$: Have gens x, y, s
$s^2=1$ $s(x)=-x$ $s(y)=-y$
$[y x] = t1 - 2cs$

Double affine Hecke algebra : algebra over $\mathbb{C}[q^{\pm1}, \tau^{\pm1}]$
generators X, Y, T : $(T-\tau)(T+\tau^{-1})=0$ and Hecke relation
$TXT = X^{-1}$, $T^{-1}YT^{-1}=Y^{-1}$
$Y^{-1}X^{-1}YXT^2=q$

- two copies (X,T) & (Y,T) of affine Hecke algebra, interchanged by a "Fourier transform" $X \to Y^{-1}$

Take $Y=e^{\hbar y}$ $X=e^{\hbar x}$ $T=se^{\hbar^2 cs}$ $q=e^{\hbar^2}$ $\tau=e^{\hbar^2 c}$
in $\hbar$-adic completion double affine specializes to our algebra $H_{t,c}$
just setting $\hbar=0$ get $(t,c)=(0,0)$ point - so we linearize in magnifying glass. $\rightsquigarrow$ $\hbar$-flat family

Relations for Cherednik algebra are very complicated in general other than for simple reflections - but in our limit get nice symplectic reflection algebra relations.

Fix (t,c), Recall $\mathfrak{h}^{reg} = \mathfrak{h} -$ root hyperplanes
For each $y \in \mathfrak{h}$ $D_y = t\frac{\partial}{\partial y} + \frac{1}{2}\sum_{\alpha \in R} c_\alpha \frac{\langle \alpha, y\rangle}{\alpha}(s_\alpha - 1) \in \mathcal{D}(\mathfrak{h}^{reg})$
(α as linear fn on $\mathfrak{h}$) Dunkl operator (rational difference, $\# W$
(Demazure - Lusztig - Dunkl - BGG -- operator cases)
(without $t\frac{\partial}{\partial y}$ -- antisymmetrize & divide by α)

Crucial property: $D_y(\mathbb{C}[\mathfrak{h}]) \subset \mathbb{C}[\mathfrak{h}]$ (doesn't introduce singularities)

Cherednik **Prop** $t\neq0$ The assignment $W \ni w \mapsto w$ $\mathfrak{h}^* \ni x \mapsto x$, $\mathfrak{h} \ni y \mapsto D_y$
extends to a well-defined faithful algebra map
$H_{t,c} \to \mathcal{D}(\mathfrak{h}^{reg}) \# W$
- i.e. rationally wrt x get "the" of $H_{t,c}$ as $\mathcal{D}(\mathfrak{h}^{reg}) \# W$

All of this lifts to double affine Hecke algebra - faithful rep by Dunkl difference operators.

So we see that $\mathbb{C}[\mathfrak{h}]$ becomes an $H_{t,c}$ module via the Dunkl representation. To define this directly, build

$\mathrm{Ind}_{S\mathfrak{h}^* \# W}^{H_{t,c}} \mathbb{1}$ (same size as $S\mathfrak{h}^*$ by PBW)

$\cong$ canonical (PBW) $\Rightarrow$ get Dunkl operators.

$\mathbb{C}[\mathfrak{h}]$

So Hecke algebra comes in roundabout way via identifying Dunkl operators

---

$\mathfrak{g} = \mathfrak{gl}_n \Rightarrow \mathfrak{h}^{reg} = \mathbb{C}^n \setminus$ diagonals

Calogero-Moser : $t\Delta_{\mathfrak{h}} + c(c+t)\sum \frac{1}{(x_i - x_j)^2} = L_{t,c}$

deformed Harish-Chandra

$\phi_t : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h}^{reg})^W$ $\qquad \frac{c}{t} \in \mathbb{CP}^1$ real param

$\mathcal{C}_{t,c}$ = Centralizer in $\mathcal{D}(\mathfrak{h}^{reg})^W$ of $L_{t,c}$
(size same as $S\mathfrak{h}^W$)

$\mathcal{B}_{t,c}$ = assoc subalgebra in $\mathcal{D}(\mathfrak{h}^{reg})^W$ gen by $\mathbb{C}[\mathfrak{h}]^W$ & $\mathcal{C}_{t,c}$.

(*) Dunkl rep $\Theta_{t,c} : H_{t,c} \longrightarrow \mathcal{D}(\mathfrak{h}^{reg}) \# W$ induces a map

$$e H_{t,c} e \longrightarrow e(\mathcal{D}(\mathfrak{h}^{reg}) \# W)e \cong \mathcal{D}(\mathfrak{h}^{reg})^W$$

Θ^{sph} - not same as just restriction of Θ to eHe

i.e. Θ image of eHe acts as diffops on invariant polys

but $\Theta(eHe) \not\subset \mathcal{D}(\mathfrak{h}^{reg})^W \subset \mathcal{D}(\mathfrak{h}^{reg})$
isom is given by projection $e - e$.

(prev. stated) **Theorem** Image$(\phi_t) = \mathcal{B}_{t,c}$

Theorem Image$(\Theta^{sph}_{t,c}) = \mathcal{B}_{t,c}$

moreover $\Theta^{sph}_{t,c} : eH_{t,c}e \xrightarrow{\sim} \mathcal{B}_{t,c}$ (isom on assoc grads ...)

so $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\phi_t} \mathcal{D}(\mathfrak{h}^{reg})^W \xleftarrow[\Theta^{sph}]{\sim} eHe$

$\Rightarrow$ map $\phi^{sph}_{t,c} := (\Theta^{sph})^{-1} \circ \phi_t : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow\!\!\!\!\rightarrow eH_{t,c}e$

2 $\phi_{t,c}^{sph}$ commutes with Fourier transform.

- Now both sides have no denominators, but in between have complicated $\mathbb{B}$ algebra with unknown poles

We have a primitive ideal $I_{t,c}$ = annihilator of $V_{c/t}$ in $U\mathfrak{g}$

$\boxed{\text{Thm}}$ $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}} / (\mathcal{D}(\mathfrak{g}) I_{t,c})^{\mathfrak{g}} \simeq e H_{t,c} e$

quantum Hamiltonian reduction

so eHe is quantum C-M space!

$t=0$ get just C-M space (classical), describes spec eHe.

$\mathbb{C}[CM] \simeq eH_{0,c}e$

Goal: "Double" rep theory of $\mathfrak{g}$

Candidate: modules for $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$ ...

better candidate: $ad: U\mathfrak{g} \to \mathcal{D}(\mathfrak{g})$, take an ideal gen by an infin. character

$J_\chi = \mathrm{Ker}(\chi: Z U\mathfrak{g} \to \mathbb{C})$ minimal primitive ideal $U\mathfrak{g} J_\chi$

$\Rightarrow$ study $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$ modules annihilated by $ad(U\mathfrak{g} J_\chi)^{\mathfrak{g}}$

- impose central character, adjointly.

Our $I_{t,c}$ is somewhat bigger in general - a degenerate version.

Irreps of $H_{0,c}$ $\simeq$ points of C-M space, $[X,Y]+Id$ has rank one.

X, Y ~ "double Langlands parameters"

Want to study irreps of $H_{t,c}$.

Suppose $t \neq 0$ (close to Weyl alg case) - expect no finite dim irreducibles. Have natural category of holonomic modules

Category $Hol(H_{t,c})$ = $H_{t,c}$-modules M with Gelfand Kirillov $\dim(M) = \dim \mathfrak{h}$

<u>DeRham functor</u>: $\mathbb{C}[\mathfrak{h}]$ is an $H_{t,c}$ module via Dunkl (think of as right module)

$DR: M \longrightarrow \mathbb{C}[\mathfrak{h}] \otimes_{H_{t,c}} M$

For any point in $\mathfrak{h}/W \ni p$ can localize $\mathbb{C}[\mathfrak{h}]$ there, action still well-defined -- Dunkl's are local up to W action.

$M \longrightarrow \mathcal{O}_p \overset{L}{\otimes}_{H_{t,c}} M$, get complex of sheaves on h/W.
Q: what is the resulting equivalence of categories?

- generalize deRham functor when we have a distinguished module of the right size.

$Hol(H_{t,c}) \supset Mod(H_{t,c} \supset \mathbb{C}[h])$ subcategory of modules which are loc fin wrt $\mathbb{C}[h]$ - supported at fin many points. Insures automatically a "reg sing" condition. - GK dim condition is then automatic

Drinfeld: "Spectrum of $H_{t,c}$ has open set we understand - $\mathcal{D}(h^{reg}) \# W$. $H_{t,c}$ has Fourier transform $\to$ get another simple open.

D-module on line can be recovered from its restriction to an open subset plus restriction of Fourier transform to some other open.

Know localization of $H_{t,c}$ $\rightsquigarrow$ know category of reps modulo a thick subcategory, same for Fourier transform.

V. Ginzburg $\boxed{\text{Double Affine Hecke Algebras}}$ 11/30/00

Def. H Hecke algebra for a finite Coxeter group W: gens T_s s simple reflections in W

i) $(T_s+1)(T_s-q)=0$

ii) for any reduced expression of $w=s_{i_1}\cdots s_{i_n}$, the elmt $T_w = T_{s_{i_1}} \cdots T_{s_{i_n}}$ indep. of reduced expression for w, and $T_w T_y = T_{wy}$ whenever $\ell(w)+\ell(y)=\ell(wy)$.
& $\{T_w : w\in W\}$ form $\mathbb{C}[q,q^{-1}]$ - basis.

· i.e. there are two defining relations, iii is not one of them but rather a characterizing property...

Affine Weyl group $\dot{W} = W \ltimes X^*(T) \supset_{\text{fin index}} W_{aff} = W \ltimes Q$

$X^*(T)$: weight lattice; $\dot{W}$: not Coxeter gp. Q: root lattice; W_{aff}: Coxeter gp

Can form Hecke algebra $H(W_{aff})$ & enlarge it to $\dot{H}$ affine Hecke alg with basis labeled by $\dot{W}$.

q prime power $\dot{H}$ comes as Hecke algebra for $G(\mathbb{Q}_p)$, Iwahori pair - in fact depending on center of G can get any thing between $\dot{H}$ & $H(W_{aff})$. Define deg using analog of length function on $\dot{W}$.

<u>Bernstein (& Zelevinsky)</u> : give presentation of $\dot{H}$, via a basis $\{T_w e^\lambda\}$ $w \in W$ $\lambda \in X^*(T)$

T_λ span $H < \dot{H}$, e^λ span $\mathbb{Z}[q,q^{-1}][X^*(T)]$ group algebra + commutation relations. ... "affinization of H" via $X^*(T)$

<u>Cherednik double affine</u> $\ddot{H}$: take $\dot{H}$ as input data in B-Z construction in place of H.

i.e. write T_w $w \in \dot{W}$, and replace $X^*(T)$ by $X_*(T)$ ... "affinization of $\dot{H}$" via $X_*(T)$ -

- use usual action of affine Weyl group on the lattice? $W \ltimes X^*(T)$ acts on $X_*(T)$ (need bilinear form)

Cherednik - IMRN (Duke), Kirillov Jr (BAMS)

The two lattices are appearing in a very nonsymmetric way....

Miracle : if we replace $_*$ by * in the process we obtain the same algebra.... <u>Fourier transform</u> property - very nontrivial!

<u>Theorem</u> X_* & X^* enter symmetrically (e.g. if self-dual lattice get automorphism of algebra exchanging two copies)

.. Unfortunately don't know a definition analogous to Coxeter def.

Kazhdan comment why quantum groups? one reason! affine Hecke alg is training ground for quantum group: $\dot{H}$ is quantum gp for G over field of one element.

$\ddot{H}$ should play same role for theory of "quantum double loop group" - training ground over $\mathbb{F}_1$.

Pass to Parshin type field - nonsymmetric but we would like a <u>symmetric picture</u> of double loops so don't get that definition

Spin Calogero is "double loop" analog of Calogero

Asymmetry of definition built into theory, but morally should relate to double affine K-M ⇝ 2d Langlands...?
- Structure theory & geometric theory exist now for double affine quantum gps, but not rep theory — precisely analogous.

$t=0$ analog of critical level... (when quantized)
- don't see an Heisenberg algebra (center always same)
CM space analog of opers (Spec of center..)

$\ddot{H}$ has two parameters q, t ...

sl_2: $(T+1)(T-q)=0$ $\quad XYX^{-1}Y^{-1}T^2 = t$ Heisenberg relation
t is Heisenberg parameter, q ~ # elements of finite field.
$TXT = X \quad T^{-1}YT^{-1} = Y$.
t is analogous to τ in $H_{\tau,c}$ $\quad (t = e^{2\pi i \tau})$
-- critical level is $t=1$.
Hope for $t=1$ for this & quantum affine get big center...

K-theoretic construction of $\ddot{H}$ (affinization of K-L, Ginzburg construction for H)

[Garland-Grojnowski def.]

$\mathcal{B}$ affine flag manifold $= \{$Iwahori subalg. $\mathfrak{b} \subset \mathfrak{g}((z))\}$
$T^*\mathcal{B} = \{$Iwahori subalgebras $\mathfrak{b} \subset \mathfrak{g}((z))$ & quasinilpotent elt of $\mathfrak{b}\}$
e.g. at $\mathfrak{b}_0 + z\mathfrak{g}[[z]] \Rightarrow \mathfrak{n}_0 + z\mathfrak{g}[[z]]$ $\quad$ $= \underline{\text{commutant of } \mathfrak{b}}$

For any $w \in W_{aff}$ ⇝ coset $Y_w \subset \mathcal{B}\times\mathcal{B}$

Drinfeld: Smoothness: see minor of Jacobian nondegen ⟷ completion is something standard ⟷ Grothendieck lifting property
OR can say finite homological dim..
in infinite situation these two diverge!

Ginzburg: $\quad T^*(\mathcal{B}\times\mathcal{B}) \supset Z$ Steinberg variety $= \coprod T^*_{Y_w}(\mathcal{B}\times\mathcal{B})$

Langlands duality flips rotation of loop & central extension.
here we don't need extension but do need rotations ---

t ⟷ rotations of loop (equivariantly) ⟷ dilation of fiber $\mathbb{C}^*$ act.

⇝ $K^{(G((z)) \rtimes \mathbb{C}^*)\times\mathbb{C}^*}(Z)$ "module over rep of ~~dil~~ rotating loop

rep of dilation $\mathbb{C}^*$

$$R(G((z))\rtimes\mathbb{C}^*\times\mathbb{C}^*) \overset{\text{formally..}}{=} R(G((z)))\otimes\mathbb{C}[t^{\pm1},q^{\pm1}]$$

R = Grot. group of reps...

fin dim case $K^{G\times\mathbb{C}^*}(Z)$ -- coherent sheaves

don't know exactly what $R(G((z))\cdots)$ should mean - in finite dim setting get just rep rings...

Alternate approach! fix $b_0\in\mathcal{B}$ restrict $\mathcal{B}\times\mathcal{B}\to\mathcal{B}\times b_0$

$p: T^*(\mathcal{B}\times\mathcal{B})\to\mathcal{B}\times\mathcal{B}\xrightarrow[pr_2]{}\mathcal{B}\ni b_0$, take inverse image

$Z\cap p^{-1}(b_0)=Z_{b_0}$ (everything G-equivariant). Iwahori I = isotropy of b_0

$I=B_0\ltimes$ pronilpotent group.

- formally replace our K group by $K^{(I\rtimes\mathbb{C}^*)\times\mathbb{C}^*}(Z_{b_0})$

but B_0 = unipotent $\cdot$ torus.. K doesn't feel unipotent groups!

So $K^I(\ )=K^T(\ )\Rightarrow K^{T\times\mathbb{C}^*\times\mathbb{C}^*}(Z_{b_0})$

$T,\mathbb{C}^*,\mathbb{C}^*$ all commute...

I orbits on $\mathcal{B}$: $IwI/I\subset G((z))/I$ finite dim, with projective closures...

but conormal directions are infinite $T^*_{\mathcal{B}_w}\mathcal{B}$..

- consider coherent sheaves which along $\mathcal{B}$ sit on finite dim pieces & along T^* direction are pullbacks from some finite quotient

- i.e. finitely presented modules over $\mathbb{C}[T^*]$.

$\to$ can define convolution algebra.

(Ga-G.) **Theorem** $K^{T\times\mathbb{C}^*\times\mathbb{C}^*}(Z_{b_0})\simeq\ddot{H}$

-- module over $R(T)$ but no longer central $R(T)$

($\ddot{H}$ generically has no center).

"Langlands dual" picture, after Kapranov: (JAMS) ~~[illegible]~~

k 1-dim local field (nonarch.) residue class field $\mathbb{F}_q$

(q will be our q.)

$K=k((t))$ Parshin 2d local field

$\cup$

max ideal $m_k\subset\mathcal{O}_K=k[[z]]$ $\quad p:\mathcal{O}_K\to\mathcal{O}_K/m=k$.

K not locally compact... $k \supset \mathcal{O}_k$

$G(K) \supset G(\mathcal{O}_K) \xrightarrow{p} G(k) \supset I$ Iwahori

$\mathcal{O} = p^{-1}(\mathcal{O}_k) \subset \mathcal{O}_K$ "regular" part... inductive definition.

~~...~~ $p^{-1}(I) \subset G(\mathcal{O}_K)$

$0 \to \mathbb{Z} \to K^*/\mathcal{O}^* \to \mathbb{Z} \to 0$ split but not canonically

two uniformizers,

$\ddot{W} = \dot{W} \ltimes (X^*(T) \otimes K^*/\mathcal{O}^*)$ $K^*/\mathcal{O}^*$ analog of $\mathbb{Z}$ for valuation in affine setting.. get two copies of same lattice $X^*(T)$ in this construction.

Correction: need central extension $0 \to k^* \to \hat{G}(K) \to G(K) \to \emptyset$, splits over $G(\mathcal{O})$ (and $G(\mathcal{O}_k)$)

"Weyl-Heisenberg" group $0 \to \mathbb{Z} \to \hat{W} \to \ddot{W} \to 1$ Heisenberg-type extension using pairing on $X^*(T) \times X^*(T) \to \mathbb{Z}$
- replace abelian part by Heisenberg, still semidirect product

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<u>Digression to affine setup</u> $\mathcal{B} = G((z))/I$

Also have semi-infinite (periodic) flags $\mathcal{B}_{per} = G((z))/(T(\mathbb{C}[[z]])N((z)))$

$B_0 = TN \subset G$ fixed Borel $\qquad T(\mathcal{O}_k)N(k)$

$\sim$ more or less $G/B((z))$.

$p: G(\mathcal{O}_K) \to G(k)$, Let $J = p^{-1}(T(\mathcal{O}_k)N(k))$ maybe extend J by $\mathcal{O}_k^*$

Kapranov's flag variety! $\hat{\mathcal{B}} = \hat{G}(K)/J$

Claim (Garland) $I \backslash \hat{G}(K)/J \overset{\text{approx}}{\longleftrightarrow} \hat{\hat{W}}$, $\boxed{I\backslash G(K)/J \leftrightarrow \ddot{W}}$

$X^*(T)$ acts on $G(K)/J$ on the right $\dashrightarrow$ get lattice in $\ddot{W}$.

$\ddot{W} = \dot{W} \ltimes L$ $\quad (L = X^*(T))$ get $\dot{W}$ by replacing $T(\mathcal{O}_k)N(k)$ by $B(k)$.

Iwahori orbits on $G(K)/J$ - consists of pieces which are "finite dim" can be exhausted by

by locally compact subsets with natural measure (each piece has infinite volume.. ~ like Lebesgue measure on k.

Let $I_{per} = T[[z]]N((z)) \subset G((z))$ $\mathcal{B}_{per} = G((z))/I_{per}$.

compact support $\mathcal{F}un_0(\mathcal{B}_{per})$ For every element w of $W_{aff} = I_{per}\backslash G((z))/I_{per}$
get intertwining operator by formal integration on coset
-- but destroys compact support
$\mathcal{F}_0 = \mathcal{F}un_0(\mathcal{B}_{per}) \xrightarrow{T_w} \mathcal{F}un(\mathcal{B}_{per})$ unspecified support.
- cannot compose this ...
(also Kazhdan-Braverman) trick to make these composable:
use action from right of L:
allow combinations $\sum_{\ell \in L} f_\ell \cdot \ell \in \widehat{\mathcal{F}un_0 \otimes \mathbb{C}[L]}$ f_ℓ compactly supported; formal completion of group algebra

For particular such sums $\quad$ $\parallel$ Completion of $\mathcal{F}un_0 \otimes \mathbb{C}[T]$

Inside $\widehat{\mathbb{C}[T] \otimes \mathcal{F}un_0}$ consider sums which sum to rational function on T (in some expansion).
- call this $\mathcal{F}_{rational}$

Theorem Operators $T_w : \mathcal{F}_0 \longrightarrow \mathcal{F}_{rat}$: can reinterpret image of T_w as infinite sum over L as cptly supported function, & these are from rational
- Intertwining operator depends rationally on parameters (local version of Eisenstein series) - intertwiners not defined pointwise but on family of principal series can make sense, & have poles

Can make $T_w : \mathcal{F}_{rat} \circlearrowleft$
$\Downarrow$
generate an algebra over $\mathbb{C}(T)$

Theorem The subalgebra in the $\mathbb{C}(T)$-algebra generated by T_w, which takes $\mathcal{F}_0$ into $\mathcal{F}_0$, is isomorphic to $\mathring{H}$.

Try to find $\mathring{H} \subset \mathbb{C}(T) \# W$ - coefficients must match on hyperplanes not to create poles.

Claim: Formulations 2 theorems generalize to double context, with no changes.

Schubert sets in 1-dim case are finite sets, whole space is locally compact. In 2-d case Schubert cells are locally cpt, but whole space is complicated.

Irreps of $\dot{H}$ $\sim$ $\left\{ \begin{array}{l} G^L \text{ conjugacy classes of data } (s,u,\chi) \\ s \text{ semisimple, } u \text{ unipotent, } \chi \text{ rep of finite group, } sus^{-1}=u^q \end{array} \right\}$

- subtle answer, hard to describe via root data.

$$= \mathrm{Hom}(\Gamma, G^L)/\mathrm{Ad}\, G^L$$

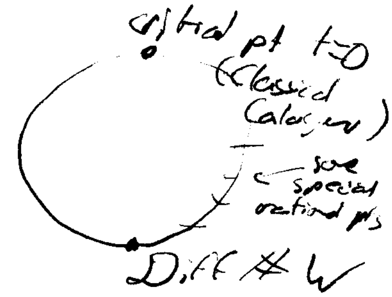
Hope - do same in two-dim setting
- in 1-d case Langlands gives hint as to answer, but in 2-d case don't have even Langlands

$s,u \leftrightarrow X,Y$ matrices

CM space analog of (s,u) space - up to some deformation... in somewhat degen limit we know both s,u
hope that understanding this variety will help get this parametrization & maybe guess 2-d Langlands.

Quantum CM for arbitrary root system - have algebra $H_{t,c}$.
Need Wilson-type compactification allowing pts to collide. Don't know reps of $H_{t,c}$

$\underline{GL_n}$ t,c up to rescaling give $\mathbb{CP}^1$



Off Interesting rational points: u,χ parameters vanish.. so to see them need to look at hard pts, otherwise see "obvious" s parameter. $D\#W$ case: our category are D-mods supported at certain special pts -- these are the s parameters
~~generated~~