V. Ginzburg - Vanishing Cycles for $\mathcal{D}$-Modules

$\mathcal{C}^\times$: all $\mathcal{D}_{\mathbb{C}^*}$-modules with all subquotients $\simeq R = \mathbb{C}[t,t^{-1}]$

Theorem : $\mathcal{C}^\times \xrightarrow[\text{monodromy}]{\sim} \{(V,T) \mid V$ vector space, $T \circlearrowright V$ unipotent$\}$

Any object in RHS is direct sum of Jordan blocks $\mathbb{C}^n, \begin{pmatrix} 1 & \ddots & 0 \\ 0 & \ddots & 1 \end{pmatrix}$

.. this is isomorphic to image of
$J_n = R \cdot \log^{n-1} t + R \log^{n-2} t + \dots + R \log t + R$
naturally $\mathcal{D}_{\mathbb{C}^*}$-module.

J_n has $\mathbb{C}$-basis $t^k \log^l t \quad k \in \mathbb{Z} \quad l = 0 \dots n-1$
$t\partial : t^k \log^l \longmapsto k t^k \log^l + l\, t^k \log^{l-1}$

For fixed k get $n \times n$ matrix of action on $t^k \log^\bullet$: $t\partial = \begin{pmatrix} k & \ddots & 0 \\ 0 & \ddots & k \end{pmatrix}$

Monodromy $T = e^{2\pi i t\partial} = \begin{pmatrix} 1 & \ddots & * \\ 0 & \ddots & 1 \end{pmatrix}$ get essential surjectivity

Fully faithful : need $\mathrm{Hom}_{\mathcal{D}}(J_n, J_m) = \mathrm{Hom}_{(V,T)}(\mathrm{Jord}_n, \mathrm{Jord}_m)$

$\mathrm{Hom}(\mathrm{Jord}_n, \mathrm{Jord}_m)$: $\rightrightarrows \rightrightarrows$ have maps if $n > m$

...

Alternative interpretation : $M \in \mathcal{C}^\times$, $t\partial_t$ acts loc finitely with integer eigenvalues $\quad M \in \mathcal{C}^\times \Rightarrow M = \bigoplus_{i \in \mathbb{Z}} M_i$
$\dim M_i < \infty$

$$\begin{tikzcd} M_{-1} \arrow[r, shift left, "t"] \arrow[loop below, "-1"] & M_0 \arrow[l, shift left, "\partial"] \arrow[r, shift left, "t"] \arrow[loop below, "0"] & M_1 \arrow[l, shift left, "\partial"] \arrow[r, shift left, "t"] \arrow[loop below, "1"] & M_2 \arrow[l, shift left, "\partial"] \arrow[loop below, "2"] \end{tikzcd}$$

all determined by subdiagram ...

t is invertible, so all $\xrightarrow{t\partial_t}$ arrows are isos, all M_i have same dimension

Category $\mathcal{C}^\times$ has a single simple object R, but no projectives - since Jordan blocks don't (need ∞ size Jordan blocks) but has a pro-object $\mathcal{E}^{\text{proj}}$ pro-projective

J_n is n-fold extension of R $\quad \begin{pmatrix} R \\ R \\ \vdots \\ R \end{pmatrix} \xleftarrow{} \text{socle}$

$J_{n+1} \twoheadrightarrow J_n = J_{n+1} / \text{socle}$, as $\mathcal{D}$-modules
$\mathcal{E}^{\text{proj}} := \varprojlim J_n$ has cosocle (top) but no bottom $\quad \begin{matrix} R \\ R \\ R \\ \vdots \end{matrix}$

- every simple sheaf has at most one projective cover, so this is unique projective.

Lemma $\mathrm{Hom}_{\mathcal{D}}(\mathcal{E}^{proj}, M) = M_0$ weight 0 piece -

$\Rightarrow$ exact functor $\Rightarrow$ $\mathcal{E}^{proj}$ projective.

Similarly can define ind-direct limit $\mathcal{E}^{ind} = \varinjlim J_n$ - injective hull of R. embeddings tautological from description

$R \log^{n-1} t \hookrightarrow R \log^{n} t \hookrightarrow \ldots \ldots \hookrightarrow \ldots$

<u>Lemma</u> $J_n \cong \mathcal{D}/\mathcal{D}(t\partial)^n$ as $\mathcal{D}_{\mathbb{C}^*}$-modules

$\log^{n-1} \longleftarrow 1$

$\mathcal{D} = \mathbb{C}[t,t^{-1}][\partial] = \mathbb{C}[t,t^{-1}][t\partial]$

So $\mathcal{D}/\mathcal{D}(t\partial)^n$ is free R-module $1, t\partial, (t\partial)^2, \ldots (t\partial)^{n-1}$

$\rightarrow$ check above is isomorphism. $\square$

$\Rightarrow$ projections $J_n \leftarrow J_{n+1}$

$\mathcal{D}/\mathcal{D}(t\partial)^n \leftarrow \mathcal{D}/\mathcal{D}(t\partial)^{n+1}$

So $\mathcal{E}^{proj} = \varprojlim \mathcal{D}/\mathcal{D}(t\partial)^n$, so $\mathrm{Hom}(\mathcal{E}^{proj}, -)$ is generalized 0-eigenspace of $t\partial$ ie 0 component.

Need to compare $\mathrm{End}(\varprojlim_n \mathrm{Jordan}_n)$ with $\mathrm{End}(\varprojlim_n J_n) = \mathrm{End}(\mathcal{E}^{proj})$

- both isomorphic as algebras to ~~polynomial rings~~ formal power series in one variable $\mathbb{C}[[u]]$: take cyclic vector to some combination of lower basis vectors. So both categories look like modules over $\mathbb{C}[[u]]\ldots$

- $\mathcal{E}^{proj} = \varprojlim \mathcal{D}/\mathcal{D}(t\partial)^n$ $\qquad \mathcal{E}^{ind} = R[\log] = \mathbb{C}[t,t^{-1}][\log]$

<u>Remark</u> Duality $\mathbb{D}$ on holonomic modules, $\mathbb{D}(R) = R$

exact functor, takes $\mathcal{E}^n$ into itself

$\mathbb{D}$ of anything is isomorphic to itself, but filtration naturally reversed.

$\mathbb{D}\mathcal{E}^{ind} = \mathcal{E}^{pro}$

$\{0\} \hookrightarrow \mathbb{C} \xhookleftarrow{j} \mathbb{C}^*$

Consider category with two simple objects, $\delta = \mathcal{D}/\mathcal{D}t$ sitting at 0, & $\mathcal{O} = \mathbb{C}[t] = \mathcal{D}/\mathcal{D}\partial$

$\mathcal{C}$ = category of $\mathcal{D}$-modules on $\mathbb{C}$ with all subquotients $\delta, \mathcal{O}$

Pushforward: $R = \mathbb{C}[t, t^{-1}]$, $j_* R = R$, as $\mathcal{D}_{\mathbb{C}}$-module extension $R = \begin{pmatrix}\delta\\ \mathcal{O}\end{pmatrix}$.

$$j_! R = \mathbb{D} j_* \mathbb{D} R = \mathbb{D} j_* R = \mathbb{D}\begin{pmatrix}\delta\\ \mathcal{O}\end{pmatrix} = \begin{pmatrix}\mathcal{O}\\ \delta\end{pmatrix}$$

$j_! R \longrightarrow j_* R$ $\quad \begin{pmatrix}\mathcal{O}\\ \delta\end{pmatrix} \to \begin{pmatrix}\delta\\ \mathcal{O}\end{pmatrix}$, $\delta \to 0$ $\quad$ must restrict to identity on $\mathbb{C}^*$

$\Rightarrow j_{!*} R = \mathcal{O}$.

In formulas: $j_* R = \mathcal{O}[t^{-1}] = \mathcal{D}/\mathcal{D}(\partial t)$

$j_! R = \mathcal{D}/\mathcal{D}(t\partial)$

How about $j_* \mathcal{I}_n$? exact functor for open embedding, so can apply level by level on extension

$$j_* \mathcal{I}_n = \begin{pmatrix}\delta\\ \mathcal{O}\\ \delta\\ \mathcal{O}\\ \vdots\\ \delta\\ \mathcal{O}\end{pmatrix} \qquad j_! \mathcal{I}_n = \begin{pmatrix}\mathcal{O}\\ \delta\\ \mathcal{O}\\ \vdots\\ \delta\end{pmatrix}$$

$\begin{pmatrix}\mathcal{O}\ \delta \cdots \delta\ \mathcal{O}\end{pmatrix}$ $\quad \begin{pmatrix}\mathcal{O}\cdots\\ \delta\end{pmatrix}$

$\nabla^{proj} := j_! \mathcal{E}^{proj}$, $\quad \Delta^{proj} = j_* \mathcal{E}^{proj}$ $\quad \begin{pmatrix}\delta\\ \mathcal{O}\\ \vdots\end{pmatrix}$

$\nabla^{ind} = j_! \mathcal{E}^{ind}$ $\quad \mathbb{D}$ $\quad \Delta^{ind} = j_* \mathcal{E}^{ind} = \begin{pmatrix}\vdots\\ \delta\\ \mathcal{O}\end{pmatrix}$

$j_! \mathcal{E}^{proj} \longrightarrow j_* \mathcal{E}^{proj}$

must be $\begin{pmatrix}\mathcal{O}\\ \delta\\ \vdots\end{pmatrix} \to \begin{pmatrix}\delta\\ \mathcal{O}\\ \vdots\end{pmatrix}$ injective, $j_{!*} \mathcal{E}^{proj} = j_* \mathcal{E}^{proj}$

$$0 \to j_! \mathcal{E}^{proj} \to j_* \mathcal{E}^{proj} \to \delta \to 0$$

$$0 \to \delta \to j_! \mathcal{E}^{ind} \to j_* \mathcal{E}^{ind} \to 0$$

<u>Prop</u> The category $\mathcal{C}$ is equivalent to the category of quiver reps $\bullet \underset{v}{\overset{u}{\rightleftarrows}} \bullet$, $(uv)^N = 0$ $\quad N \gg 0$.

$$\begin{tikzcd} \delta \arrow[r, bend left, "u"] & 0 \arrow[l, bend left, "v"] \end{tikzcd}$$

Two proofs !

1. have two simple objects ⇒ two pro-projective covers $\Delta^{proj} \to \delta$ $\nabla^{proj} \to 0$

$M \in \mathcal{C} \longmapsto Hom(\Delta^{proj}, M)$ $Hom(\nabla^{proj}, M)$

two maps ∇^{proj} Δ^{proj}

$\frac{\delta}{0}$ $\frac{0}{\delta}$

semicircle like has as necklaces

eventually get zero on the two necklaces naturally

Any object $M = \bigoplus_{i \in \mathbb{Z}} M_i$ eigenvalues of $t\partial_t$

$$\xrightarrow{t} \boxed{M_{-1} \underset{\partial}{\overset{t}{\rightleftarrows}} M_0} \underset{\partial_t}{\overset{t}{\rightleftarrows}} M_1 \rightleftarrows \cdots$$

Claim: all maps outside the box are isomorphisms !

$t \circ \partial = t\partial$, $\partial \circ t = t \circ \partial + 1$

- matrices look like $\begin{pmatrix} 1 & & \\ & \ddots & a \\ 0 & & \ddots \end{pmatrix}$ invertible except at 0 step, for $\rightleftarrows$ $t\partial_t$

in other order get isos outside -1 step.

-- so get all info given by M_0 & M_{-1} :

$M = 0$ is all in degree zero , $M = \delta$ has only $M_{-1} = \mathbb{C}$

$$\underbrace{0 \; (1 \quad t \quad t^2 \quad t^3 \quad M_0 = \mathbb{C}}_{0} \qquad \underbrace{\begin{matrix} \partial^2\delta & \partial\delta & \delta) & 0 \\ -3 & -2 & -1 & \end{matrix}}_{\delta}$$

$\Rightarrow Hom(\Delta^{proj}, M) = M_0$

$Hom(\nabla^{proj}, M) = M_{-1}$.

$$\mathcal{C} \simeq (\underset{Hom(\Delta^{proj}, M)}{M_{-1}} \underset{u}{\overset{v}{\rightleftarrows}} \underset{Hom(\nabla^{proj}, M)}{M_0}) \qquad (uv)^N = 0$$

Quasi inverse : from $F \underset{u}{\overset{v}{\rightleftarrows}} E$ construct

$\mathcal{D}$-module $M = \mathbb{C}[t] \otimes_{\mathbb{C}} E + \mathbb{C}[\partial] \otimes_{\mathbb{C}} F$ with action

$t \cdot (1 \otimes f) = 1 \otimes v(f)$ $f \in F$

$\partial(1 \otimes e) = 1 \otimes u(e)$ $e \in E$

Relation to duality $\mathbb{D} : \mathcal{C} \circlearrowleft$

$\mathbb{D} = R\mathrm{Hom}_{\mathcal{D}}(-, \mathcal{D})$

Apply to free resolution of our M $\quad 0 \to N \xrightarrow{p} N \xrightarrow{a} M \to 0$

$N = \mathcal{D} \otimes_{\mathbb{C}} E \oplus \mathcal{D} \otimes_{\mathbb{C}} F \xrightarrow{a} M$ action map

$P : (a \otimes e, b \otimes f) \longmapsto (a\partial \otimes e, -a\,u(e)) + (-b\,v(f), bf \otimes f)$

$\mathrm{Hom}_{\mathcal{D}}(\mathcal{D} \otimes E, \mathcal{D}) = E^* \otimes_{\mathbb{C}} \mathcal{D}$ so calculate complex

$0 \to N^* \xrightarrow{p^*} N^* \to 0 \qquad N^* = \mathrm{Hom}_{\mathcal{D}}(N, \mathcal{D})$

$p^*(e^* \otimes a, f^* \otimes b) \longmapsto (e^* \otimes \partial a, -v^*(e^*) \otimes a) +$
$(-u^*(f^*) \otimes b, f^* \otimes fb)$

$\Rightarrow$ dual to $F \underset{u}{\overset{v}{\rightleftarrows}} E$ is $\quad E^* \underset{-v^*}{\overset{u^*}{\rightleftarrows}} F^*$

On category $\mathcal{C}$ we have $\mathcal{C}$ we have defined two functors

$\mathcal{C} \underset{\Phi}{\overset{\Psi}{\rightrightarrows}} \mathrm{Vect} \qquad \Psi(M) := \mathrm{Hom}(\nabla^{\mathrm{proj}}, M) = M_0$

$\Phi(M) := \mathrm{Hom}(\Delta^{\mathrm{proj}}, M) = M_{-1}$

$\psi(M)$: remove all δ's from M, take 1 dim vector space ... ie $\psi(M) = M|_{\{t=1\}}$ nearby fiber

ϕ : for similar interpretation take Fourier transform, count number of δ's.

$\psi(M) \underset{\mathrm{var}}{\overset{\mathrm{can}}{\rightleftarrows}} \phi(M) \qquad$ can = canonical
var = variation,

composite $T - \mathrm{Id} : \psi(M) \xrightarrow{\mathrm{can}} \phi(M) \xrightarrow{\mathrm{var}} \psi(M)$

$T = \exp(-2\pi i f \partial)|_{M_0}$ is monodromy map

Canonical map will be exactly u, $\qquad T - \mathrm{Id} = \mathrm{var}(\mathrm{can})$

Variation $\quad \mathrm{var} = \underbrace{\dfrac{e^{2\pi i (vu)} - 1}{(vu)}} \cdot v$

take hol fn $\dfrac{e^z - 1}{z}$ & plug in operator vu into its Taylor series.

Δ^{ind}: $R[\log] = \mathbb{C}[t,t^{-1},\log t]$ – What is meaning?

$t^n = e^{2\pi i n z}$, $\log t = z$ $\qquad$ $z \downarrow t = e^{2\pi i z}$, $\mathbb{C}^*$

So upstairs we're looking at polynomials in t & e^t which are what we need to solve constant coeff diff eqs $P(\partial)f=0$

So can define $\psi(M) = \mathrm{Tor}^{\mathcal{D}}_0(M,\Delta^{ind}) =$

$$= \{(\tilde e,\tilde f) \in E\otimes R[\log] + F\otimes R[\log] \mid \partial\tilde e = -v(\tilde f),\ t\tilde f = u(\tilde e)\}$$

t is invertible on Δ^{ind} so '' $\longleftrightarrow t\partial\tilde e = -vu\tilde e$

Solve! $\tilde e = \exp(-vu\log t)\cdot e \quad \forall e\in E$, $\tilde f = \frac{1}{t}u(\tilde e)$

$$= \frac{1}{t}\exp(-uv\log t)u(e)$$

On $\mathbb{C}^*$ have standard $\mathcal{D}$-modules $J_n \rightleftarrows J_{n+1}$
Want now an ho-proj limit!

Introduce new variable s ($\longleftrightarrow \lambda$ in b-function theory)

$$0 \to \mathbb{C}[[s]] \to \mathbb{C}((s)) \to \mathbb{C}((s))/\mathbb{C}[[s]] \to 0$$

$$\mathbb{C}((s)) \xrightarrow{\mathrm{Res}} \mathbb{C}$$

$\mathrm{Res}: \mathbb{C}[[s]] \otimes \mathbb{C}((s))/\mathbb{C}[[s]] \to \mathbb{C}$
nondegenerate

Apply to $R = \mathbb{C}[t,t^{-1}]$ & add formal symbol t^s

$$0 \to Rt^s\,\mathbb{C}[[s]] \to Rt^s\,\mathbb{C}((s)) \to Rt^s\,\mathbb{C}((s))/Rt^s\,\mathbb{C}[[s]] \to 0$$

Crucial Lemma There is a $\mathcal{D}_{\mathbb{C}}$-module isomorphism (diff wrt t)

$$Rt^s((s))/Rt^s[[t]] \simeq R[\log]$$

Pf/Construction: $t^s = e^{(\log t)s} = \sum \frac{s^i}{i!}(\log t)^i$

Map $\underset{Rt^s((s))}{\sum a_i s^i t^s} \longmapsto \mathrm{Res}_s \sum_{i,j} a_i \frac{s^i s^j \log^j t}{j!} \in R[\log]$

$Rt^s[[s]]$ maps to zero under residue since formula for t^s is purely positive ... ∎

<u>Corollary</u> $\Delta^{ind} = j_* R[\log t] = Rt^s((s)) / Rt^s[[s]]$
$= j_* \varinjlim_n J_n$

$J_n \leftrightarrow$ poles of order $\leq n$ in $Rt^s((s))$ mod reg br.

<u>Corollary</u> $J_n \simeq Rt^s[[s]] / s^n Rt^s[[s]]$

$\mathcal{E}^{proj} = \varprojlim_n J_n = Rt^s[[s]] \Rightarrow \Delta^{proj} = j_* \mathcal{E}^{proj} = Rt^s[[s]]$

So have natural extns $0 \to \Delta^{proj} \to \_\_\_\_ \to \Delta^{ind} \to 0$

$\nabla^{ind} = j_! \mathcal{E}^{ind} =$ $\qquad\qquad$ $\Delta^{ind} = j_* \mathcal{E}^{ind} =$

$\nabla^{proj} = j_! \mathcal{E}^{proj} =$ $\qquad\qquad$ $\Delta^{proj} = j_* \mathcal{E}^{proj}$

$$\boxed{j_! Rt^s((s)) \xrightarrow{\sim} j_* Rt^s((s))}$$

<u>General Case</u>
(assume X affine)

$$\begin{tikzcd}
Y = f^{-1}(0) \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] & U = X \setminus Y \arrow[l, hook', "j"'] \arrow[d] \\
\{0\} \arrow[r, hook] & \mathbb{C} & \mathbb{C}^* \arrow[l, hook']
\end{tikzcd}$$

N_U holonomic $\mathcal{D}_U$-module

$\mathrm{Image}(j_! N_U \longrightarrow j_* N_U) =: j_{!*} N_U$.

(*) <u>Proposition</u>: Let $N_0 \subset N$ be a f.dim generating subspace, $\mathcal{D}(U) \cdot N_0 = N_U$.

<u>Claim</u> $j_{!*} N_U = \mathcal{D}(X)(f^k N_0)$ for any $k \gg 0$.

(submodule of $j_* N_U$, $f^{k+1} N_U \subset f^k N_U$: decreasing chain, must stabilize)

<u>Proof</u> Let $M = j_{!*} N_U$. $\quad M|_U = N_U \Leftrightarrow N_U / M$ supported on Y.

$N_0 \subset N_u$ so $f^k N_0$ vanishes in N_u/M $k >> 0$.
i.e. $f^k N_0 \subset M$, hence $\mathcal{D}(x) f^k N_0 \subset M$.
Conversely: take $M/\mathcal{D}(x) f^k N_0$, equal to 0 on U

but M has no quotients supported off U! ∎

(Homework. Write connection underlying $\mathcal{J}_n$ ⇒ see it is self-dual:
$\mathrm{Hom}_{\mathcal{O}}(\mathcal{J}_n, \mathcal{O}) \simeq \mathcal{J}_n$ as connections.
. What corresponds to $j_* \mathcal{J}_n$, $j_! \mathcal{J}_n$ under quiver $(M_1 \rightleftarrows M_0$)

$t \longmapsto f$ $\quad t^s \longmapsto f^s$ $\quad \log t \longmapsto \log f$.

On U have $\mathcal{J}_n(f) = f^* \mathcal{J}_n = \mathcal{O}_U \log^{n-1} f + \dots + \mathcal{O}_U \log f + \mathcal{O}_U$

$\varinjlim \mathcal{J}_n(f) = \mathcal{O}_U[\log f] = \mathcal{O}_U f^s((s))/\mathcal{O}_U f^s[[s]] = f^* \Delta^{ind}$
$f^* \Delta^{pro} = \mathcal{O}_U f^s[[s]]$

Lemma on b-function If M_U is holonomic on U & $M_U = \mathcal{D}_U M_0$
M_0 f.dim generating subspace $\Rightarrow \exists b \in \mathbb{C}[s]$
s.t. $b(s) f^s M_0 \subset \mathcal{D}_X[s](f^{s+1} M_0)$

Proposition For any holonomic M_U on U we have an isomorphism
$j_!(M_U f^s((s))) \xrightarrow{\sim} j_*(M_U f^s((s)))$
(in ind-pro category...

Pf Surjectivity: image $j_{!*} M_U f^s((s)) \hookrightarrow j_* M_U f^s((s))$
By Proposition (∗) can compute $j_{!*} = \mathcal{D}_X(f^k \cdot \text{generating subspace})$ $k >> 0$

$M_0 \subset M_U$ f.d. generating subspace over $\mathcal{D}_X$
$j_{!*} = \mathcal{D}_X \cdot M_0 f^s f^k((s))$ extend scalars to $\mathbb{C}((s))$
$= \mathcal{D}_X M_0 f^{s+k}((s))$
By b-fn lemma $\supset \mathcal{D}_X b(s+k-1) M_0 f^{s+k-1}((s))$
$\supset \underbrace{b(s+k-1) b(s+k-2) \cdots\cdots}_{B(s)} \mathcal{D}_X M_0 f^s((s))$
$= B(s) j_* M f^s((s)) = j_* M f^s((s))$ ∎

Let $\mathcal{E}_f^{proj} = \varprojlim \mathcal{I}_n(f) = f_u^* \mathcal{E}_f^{proj}$

Lemma On U we have $\mathbb{D}\, Mf^s[[s]] = (\mathbb{D}M) f^s((s))/\mathcal{D}_u f^s[[s]]$
$\mathbb{D}(\mathcal{E}_f^{proj} \otimes_u M) = \mathcal{E}_f^{ind} \otimes_{\mathcal{O}_u} \mathbb{D}M$

$f^* \mathcal{I}_n$ is free rank n $\mathcal{O}_u$-module, so $f^* \mathcal{I}_n \otimes_{\mathcal{O}_u} M \sim M^n$ but as $\mathcal{D}$-module is iterated extension of M's.

Step 1 $\mathbb{D}(\mathcal{E}_f^{proj}) = \mathcal{E}_f^{ind}$

Step 2 take finite free resolution $P^n \to \dots \to P^1 \to P^0 \to M$

$\mathbb{D}M \cong \{ Hom_{\mathcal{D}_U}(P^n, \mathcal{D}_u) \leftarrow \dots Hom_{\mathcal{D}_u}(P^0, \mathcal{D}) \}$

Same resolution for $\mathcal{E}_f^{proj} \otimes M$ ($\mathcal{E}_f^{proj}$ $\mathcal{O}$-flat)

$Hom_{\mathcal{D}}(\mathcal{E}_f^{proj} \otimes_{\mathcal{O}_U} P^i, \mathcal{D}_U) = (\mathcal{E}^{proj})^\vee \otimes_{\mathcal{O}_u} Hom_{\mathcal{D}}(P, \mathcal{D})$ -

$\Downarrow$ free

$\to$ get $\mathbb{D}M$. ∎

To show $j_! \to j_*$ injective, just dualize surjectivity statement: by above lemma can regroup tensor products on open part. ∎

Corollary $j_!(Mf^s[[s]]) \longrightarrow j_*(Mf^s[[s]])$ is injective. (subs of $((s))$ version.

Def The nearby cycle functor $\psi : Hol_{\mathcal{D}_u} \longrightarrow (Hol_{\mathcal{D}_X})_{f^{-1}(0)}$ is $\psi(M) = j_* Mf^s[[s]] / j_! Mf^s[[s]]$ with canonical $\mathcal{D}$-module endomorphism, multiplication by s.

M holonomic $\mathcal{D}_u$-module, M_0 f.dim with $\mathcal{D}_U \cdot M_0 = M$

Lemma a. $j_!(Mf^s[[s]]) = \mathcal{D}_X[[s]](f^{s+k} M_0)$ as $\mathcal{D}_X[[s]]$-module $k >> 0$

b. $j_*(Mf^s[[s]]) = \mathcal{D}_X[[s]](f^{s-k} M_0)$ $k >> 0$

a: $j_{!*} = j_!$ by Corollary, so this gives !* description

b follows by duality.

Proposition $j_* M = \dfrac{\mathcal{D}_X[s](f^{s-k} M_0)}{s\,\mathcal{D}_X[s](f^{s-k} M_0)}$ (divide by any f ... b-function)

$$j_! M = \frac{\mathcal{D}_X[s](f^{s+k} M_0)}{s\,\mathcal{D}_X[s](f^{s+k} M_0)}$$

Pf On U: $0 \to M_U f^s[[s]] \xrightarrow{s\cdot} M f^s[[s]] \to M \to 0$

$j_*, j_!$ are exact $\Rightarrow$ $0 \to j_! M f^s[[s]] \xrightarrow{s} j_! M f^s[[s]] \to j_! M \to 0$

$$\overset{\shortparallel}{\mathcal{D}_X[[s]] f^{s+k} M_0} \text{ by lemma}$$

- $j_!$ doesn't embed in j_* but after adding $[[s]]$ it does! ∎

Theorem $\Psi(M)$ is holonomic, supported on $f^{-1}(0)$ and Ψ is exact! (unlike $*, !$ restricted to divisor... restriction is not exact!)

Proof of exactness General nonsense lemma:
Let F, G be exact functors $F, G : \mathcal{C} \to \mathcal{C}'$
& have in addition natural transformation $F \to G$.
which is always injective $F(M) \hookrightarrow G(M)$.
Then $M \mapsto G(M)/F(M)$ is an exact functor.

Heuristically obvious! in derived category cone $F \to G$ is automatically exact, as complexes, but know where it's concentrated!

Prop $\exists!$ lattice L in $\mathcal{M}f^s[s]$ s.t. $\forall$ real parts of all generalized eigenvalues of s acting on L/fL are in $(-1, 0]$

(View $\mathcal{M}f^s[s]$ as $\mathcal{D}_X[s,f]$-module.)

Remark Eigenvalues of s in L/fL : $\exists b \in \mathbb{C}[s]$ $b(s)(L/fL)=0$ eigenvalues $=$ roots

$sf = f(s+1) \Rightarrow$ if $\mathrm{Spec}_L(s)$ = set of roots of b = eigenvalues of s on L/fL. Then $\mathrm{Spec}_{fL}(s) = \mathrm{Spec}_L(s)+1$ shift by 1 when pass to fL/f^2L ...

Condition above $\overset{\text{on } b}{\Leftrightarrow}$ (minimal polynomial) $\forall k \in \mathbb{Z}$ $b(s-k)f^kL \subset f^{k+1}L$

Proof of proposition Existence: Start with some lattice L, let b = corresponding polynomial (generator of principal ideal annihilating L/fL). Roots of b may be assumed to be ≤ 1 in real part by multiplying by power of f.

Suppose b has factors $s-\lambda$ & $s-(\lambda-1)$
$\Rightarrow$ write procedure to shift to right into our interval:

Write $b(s) = b_1(s)\, b_2(s)$ construct new lattice:

$$L' = f\cdot L + \overset{s-(\lambda-1)}{b_1(s)}\, L \implies b_1(s-1)b_2(s) \text{ annihilates } L/fL \quad \ldots \text{ iterate this move all roots into } (-1,0].$$

__Uniqueness__ Suppose $b(s)L \subset fL$, $b'(s)L' \subset fL'$ and $\mathrm{Spec}_L(s)$, $\mathrm{Spec}_{L'}(s) \subset (-1,0]$.

By general properties of lattices $\exists N \geq 0$ s.t. $L \subset f^{-N}L'$.

$$\Rightarrow b(s-k)f^kL \subset f^{k+1}L \subset f^{k+1-N}L'$$
$$b'(s-k+N)f^kL \subset b'(s-k+N)f^{k-N}L' \subset f^{k+1-N}L'$$

If $N>0 \Rightarrow b(s-k)$ & $b'(s-k+N)$ have no roots in common $\rightsquigarrow$ deduce from above $L \subset f^{-N+1}L'$, iterating $\Rightarrow$ eventually $L \subset L'$. $\blacksquare$

Last time: L/tL is holonomic, so b exists $\Rightarrow$ total Ψ functor.

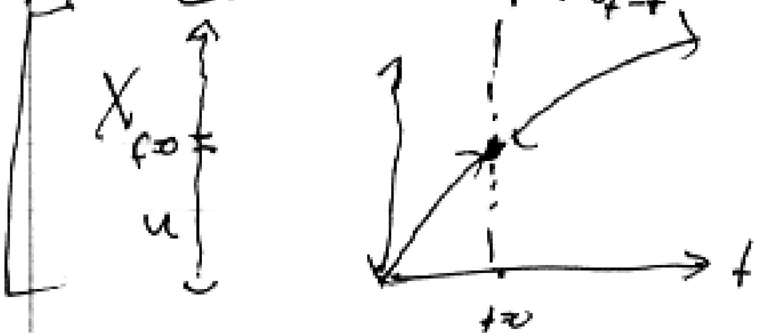
Malgrange construction: X, f fn on X

$X \xhookrightarrow{i} X\times\mathbb{C}$ graph of f, smooth subvar, $t\in\mathbb{C}^1$

M a $\mathcal{D}_X$-module $\quad U = X \setminus f^{-1}(0)$

Proposition: Given M define a map $j_*Mf^s[s] \longrightarrow i_*M$,

$m\cdot f^s \longmapsto m\,\delta_{f-t} = \left[\frac{m}{f-t}\right]$, $\quad s \longmapsto -t\partial_t$



$U \leftrightarrow$ set $t\neq 0$ on graph

Then this is a $\mathcal{D}_X$-map & map $t: m\otimes f^s \mapsto m(s+1)f^{s+1}$ corresponds to mult by t, $\quad s \leftrightarrow -t\partial_t$.

So $\mathcal{D}_X[s,t,t^{-1}] = \mathcal{D}_X[t\partial_t, t, t^{-1}] \subset \mathcal{D}_{X\times\mathbb{C}}$

Choosing a lattice $\leftrightarrow$ choosing $\mathcal{D}_X[t, t\partial_t]$ submodule $L\subset i_*M$.

L/tL ~ restrict to $t=0$.

Lattice L: preferred extension to $t=0$, get $\mathcal{D}$-mod on $X = X\times\{0\}\subset X\times\mathbb{C}$, $\Psi(M)$.

$f^{-1}(0)=Y \xrightarrow{i} X \xhookleftarrow{j} U$

For $\mathcal{O}$-modules have well defined specialization

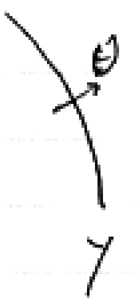
sp: $K(U) \longrightarrow K(Y)$

$\mathcal{F} \dashrightarrow$ choose lattice L in $X \dashrightarrow i^*L$.

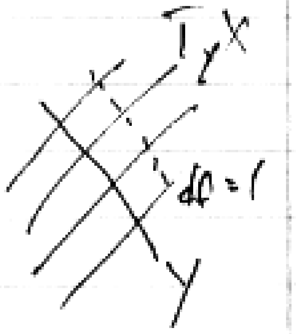
get well defined composite on K group.

$\mathcal{D}$-modules: can do much better, canonical lattice, can do on level of $\mathcal{D}$-modules directly .... $\Rightarrow \Psi$.

1-dim case: $M = \oplus M_i \qquad M_{-1} \rightleftarrows M_0$



Assume now that $df \neq 0$ on Y $\Rightarrow$ Y smooth.
Then f gives local transversal to Y. get vector field (locally) θ "$t\partial_t$" in transverse direction... unique only up to second order... Not acting locally finitely
Take normal bundle $T_Y X$ with Euler field $E_u = t\partial_t$



Verdier: re-interpret Ψ as specialization functor $Sp_{X/Y}$
(for any smooth submanifold) exact functor
$Sp_{X/Y} : \mathcal{H}ol_u \longrightarrow \mathcal{H}ol^{mon}_{T_Y X \setminus Y}$ monodromic: loc finite wrt Euler field
If $Y = f^{-1}(0)$, $df \in T_Y^* X$. $\Rightarrow$ look at section $df = 1$ in $T_Y X$, get $\mathcal{D}$-module on X, our old Ψ.

Y submanifold with ideal sheaf $I_Y \subset \mathcal{O}_X$ $\Rightarrow$ deform X to $T_Y X$.
$Sp_{X/Y}(\mathcal{M}) = \Psi_t(\mathcal{M} \otimes \mathbb{C}[t, t^{-1}])$ $t : \mathcal{X} \longrightarrow \mathbb{C}$, $t^{-1}(0) = T_Y X$, $T^{-1}(\underset{\neq 0}{\lambda}) = X$
for $\mathcal{M}$ $\mathcal{D}_X$-module.