

# T. Hales - Motivic Integration & Applications

10/24/04

p-adic integral

$$\int_{O_F} |x^n| dx \quad F = p\text{-adic field} \quad O_F = \text{integers}$$

$q = \text{max. order field } k \text{'s order}$

$$= \frac{q^n(q-1)}{q^{n+1}-1} \quad \text{independent of } F! \quad (\text{almost, get rid of } q)$$

$$q = |\mathbb{A}'(k)| \rightarrow L = [\mathbb{A}'_k] \quad \text{Lefschetz motive}$$

$$\int_{k[[t]]} |x^n| dx = \frac{L^n(L-1)}{L^{n+1}-1} \quad k \text{ any field}$$

Replace p-adic integrals by such symbols, wherever counting points...

In general symbol  $[X]$  for each variety/k

$$[X] = [X \setminus Y] + [Y] \quad Y \subset X \text{ closed}$$

$$[X](Y) = [X \setminus Y]$$

& complete on  $(\sum_0^{\infty} L^{-(n+1)k} \text{ should be in our ring})$

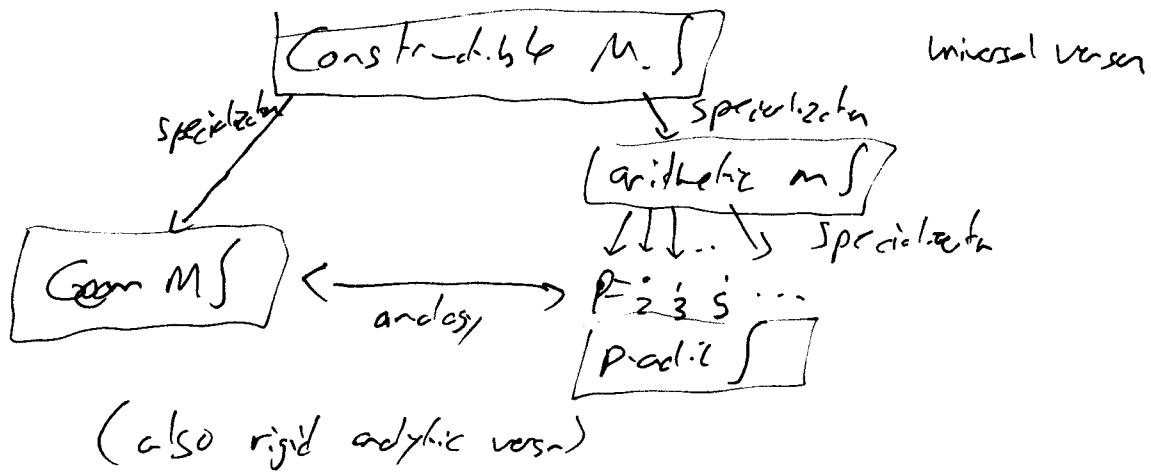
$$\Rightarrow K_0(\text{Var}_k)[L^{-1}]$$

Theory born 12/7/95 Kontsevich, Doref-Looijer 96-99

Geometric motivic integration:  $X(O_F) \rightsquigarrow X(C[[t]])$

2000-2003 arithmetic motivic integration (Doref-Looijer)

2004 - ... constructible motivic integration (Cluckers-Looijer)



Arithmetic ms : answer  $\in$  completion of K-garoud  
 Chow varieties (not Lefschetz)

Objects: 1st order formulas " $x^2+y^2=1$ "

" $\exists y: x^3=y$ " rather than sets of solutions -  
 volume of a formula, rather than its solutions.

-- Boolean combinations of formulas

can interpret formula in different rings, what universal arith.

logical symbols:  $\forall \exists \vee \wedge \neg$  (negation) =

val  $\begin{cases} p\text{-adic symbols: } 0, 1, +, \times, \neq; \\ \text{angular component} \\ \text{residue field: } 0, 1, +, \times, \xi; \quad (\text{leading coeff}) \\ \text{Valued Sp: } 0, +, \leq, m; \quad (\text{no product here: avoid undecidability results}) \end{cases}$

Ex:  $SL_n(\mathbb{Q})$ :  $X=(x_{ij})$   $\det X=1$ ,  $\text{val } x_{ij} \geq 0$

Ex: conjugacy class of  $X=(x_{ij})$ :  $\exists g$  s.t.  
 $g X g^{-1} = \begin{pmatrix} I_n & 0 \\ 0 & \square \end{pmatrix}$

Quantifier elimination:  $\exists X : x^2+ax+b=0$

$\iff a^2-4b = \square$  (almost all p)

$\iff \text{val}(a^2-4b) \equiv 0 \pmod{2}$

$$\exists \xi, \xi^2 = a(a^2-4b)$$

Replace parity quantifiers by residue field quantifiers.

1. Use quantifier elim to remove param. quantifiers  
 Presburger  $\Rightarrow$  remove integer quantifiers  
 Galois stratifications  $\Rightarrow$  " residue field "

So end up with constructible sets in alg. geometry

2. resulting formulae are  $\varphi_m : \mathbb{A}^N(k[[t]])^{mod t^M}$   
 $\lim_{m \rightarrow \infty} \frac{[\varphi_m]}{\|m\|} = \text{arithmetic value of } \varphi.$

[Uniform quantifier elimination : indep of underlying field.  
 Also in characteristics up to rational equivalence  
 answer indep of elimination procedure ]

$$\begin{array}{ccc}
 \varphi & \xleftarrow{\quad} & \xrightarrow{\quad} \\
 \left\{ \begin{array}{l} \varphi \\ \text{Arith} \end{array} \right. & & \text{vol}\left(\{x \in \mathbb{A}^N / \varphi^F(x)\} \cap x \right) \\
 \text{trace of} & \downarrow & \\
 F \otimes \mathbb{Z}_p & & F = \mathbb{Z}_p \\
 n_p & \cancel{\quad} \quad \cancel{\quad} & \text{almost all } p \text{ (Denef - Loger)}
 \end{array}$$

### Constructible w/

- integrals can have parameters
- " over  $k[[t]]$  not just  $k[[t]]$
- $X_{k[[t]]} \times X_k \times \mathbb{Z}^n$  not just  $X(k[[t]])$
- first order formulas
- no completions of rings required!  
 ("all geometric series are summed")  
 (rational function note)

formulas of subassignments: subsets of parts of  
 $X_k(\mathbb{F}) \times X_k \times \mathbb{Z}$  over any field extensions, act nec.  
subfunctors.

$\text{Int}_\alpha$ : integration =  $\int$  over  $\mathbb{C}$ -bars.

-- 1 dim at a time. + Fubini

Quantifier elimination breaks space into cells, on which  
 truth value is constant (P.J. Cohen, Doret, Pas, Collins..)  
 -- if can integrate over cells, can integrate in general.

### Applications (pre-constructible mf)

1. Lifting orbital integrals from char  $p$  to char 0.

-- Fundamental Lemma (GKM, Laumon-Ngo)  
 - i. <sup>general description</sup> writing goes, ~~assume~~

Can we lift this to char 0?

mf works in all char, answers independent of field  
 → arguments for almost all  $p$ ,  $\implies$  (by global arguments)  
 get full statement.

Laumon-Ngo prove for unitary goes in char  $> 0$ .

Hales-Cunningham:

Theorem Let  $\psi(x, s)$  locally const formula,  
 .  $S \subset \mathbb{A}^m / \mathbb{Z}[1/\ell]$  affine  
 . assert projects down to formula only in  $\{s\}$  variables.  
 (ie parameters over residue field)  
 $\implies$  volume depends on  $F$  only through  $\mathbb{F}_q$ .  
 (get formula for volume)

So whenever fundamental lemma given by such identity  $\Rightarrow$   
 indep of field.

Problems: integrals of fund. lemma depend on  $p$ -adic parameters  $\rightarrow$  mult. by  $\pi$  the integrals depend on conjugacy class only through orders mod  $p$ .

Cor For good elts in classical groups  $\Rightarrow$  fund. lemma in pos. char implies it in char 0.

Waldspurger: F.L. in pos. char  $\Rightarrow$  F.L. in char 0  
(2004) in complete generality

2. (J. Gordon) Characters of Reductive Groups.

$G = \mathrm{Sp}(2n)$  or  $\mathrm{SO}(2n+1)$

$R_{T,x}^{G^F}$  Deligne-Lusztig character

$T = T_w$  elliptic, good poster  $\Rightarrow R_{T,x}^{\text{crys}}$

(reps that exist for any field)

$\mathrm{Ind}_k^{G(F)} R_{T,x}$  rep of  $G(F)$ ,  $F$   $p$ -adic, refld  $T_w^F$

$\oplus_w$  character on  $V$  topologically unipotent set

Theorem (Gordon) Let  $\Gamma$  be a char function of a locally  $cast$  definable set  
 $\Rightarrow$   $\exists$  virtual char motive  $M_{\alpha,w}$  calculating the character distribution on  $\Gamma$  for almost all  $p$ .

Fundamental Problem

$k$  field char 0,  $S$  definable sub- $\mathbb{A}^m$  "  $\sum \mathbb{A}^n \times \mathbb{A}^m \otimes_{k(\mathbb{A})} k$ "  
 $\mathbb{Z} \xrightarrow{f} S$  cl. morphism

Clarkas-Lusztig:

abelian semigroup of positive constructible fns on  $\mathbb{Z}$ ,  $S$

$$I_S G(z) \subseteq G(z)$$

$$\int f, \\ G(S)$$

$I_S$  =  $S$ -integrable functions

(4)  $\in$  Rep theory, defined on reg SS elts of  
any reductive Lie algebra / local fields

Ex. Steinberg character, Shalika genus for  
nilpotent orbits, characters of distinguished cuspidal reps  
Fourier transforms of nilpotent orbits,  
some orbital integrals, or for apertures in flags

Problem/Conjecture: for each such (4)  $\exists$  constructible  
function  $\mathcal{G}_{\text{char}}$  on reg semisimple lattices  
s.t almost all p specializes to actual p-adic value.  
- ie all basic objects of p-adic rep theory are  
geometric, constructible.