UCSB
7/03

M. Hopkins · Algebraic Topology & Differential Forms 7/14/03

Joint w/ Singer, Freed

$\Sigma \times S$ family of Riemann Surfaces over S
$\alpha \in H^3(\Sigma \times S)$

$\int_\Sigma \alpha \in H^1(S;\mathbb{Z})$ = homotopy class of maps $S \to S^1$

Might want to use for Lagrangian etc - need to refine from homotopy class to actual map ...
Speak of more refined invariants, mixing forms & topology ...

Differential Functions

M smooth manifold, X topological space + choice of real cocycle $\iota \in Z^n(X;\mathbb{R})$

A differential function $M \to (X,\iota)$ is a triple (c,h,ω) :
$c: M \to X$ map
$h \in C^{n-1}(M;\mathbb{R})$
$\omega \in \Omega^n(M)$ s.t. $\delta h = \omega - c^*(\iota)$

Singular cochains h pretty discrete / combinatorial
- but we want to make such functions into a space, talk about families $\longrightarrow$
take formal / combinatorial approach to spaces...

Combinatorial model for spaces:
Y space, capture most topological invariants of Y from the collection of sets:

$\mathrm{Sing}_0 Y = \{$ points of $Y \}$
$\mathrm{Sing}_1 Y = \{ \Delta^1 \to Y \}$ paths in Y
$\mathrm{Sing}_2 Y = \{ \Delta^2 \to Y \}$
...

just as sets

tells us how to map any simplicial complex into Y
- given structure of patching
— ie as simplicial set
describe space combinatorially in terms of its simplices

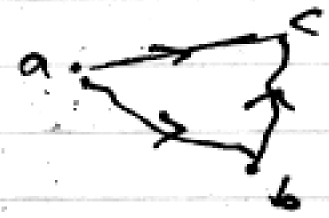
Differential function space: $(X,z)^M$ with n-simplices given by functions $(c,h,\omega) : M \times \Delta^n \longrightarrow (X,z)$

No new info here beyond ordinary function space --
extra info: have a filtration on this space:

$\mathrm{filt}_k (X,z)^M$ = space with n-simplices $(c,h,\omega) : M \times \Delta^n \to (X,z)$
s.t. $\omega \in \bigoplus_{i \le k} \Omega^i(M) \otimes \Omega^j(\Delta^n)$

filtration by Künneth components in Δ^n.

$\Pi_{\le 1}$
Fundamental groupoid -- introduced by Reidemeister
Objects = points
Morphisms = paths
composites = 2-simplices



Automorphisms = π_1

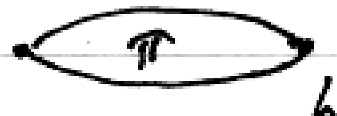
e.g. $X = \mathbb{CP}^\infty = BU(1)$, $z \in Z^2(\mathbb{CP}^\infty, \mathbb{R})$
representing c_1:

Study $\Pi_{\le 1}\, \mathrm{filt}_2 ((X,z)^M)$:

Objects = $U(1)$-bundles over M, [with connection]
$(c,h,\omega) : M \longrightarrow (X,z)$ $\quad \omega \in \Omega^2(M)$
ω represents c_1 of bundle [$\rightsquigarrow$ curvature]
[h encodes connection form]

[integrality properties are captured in fact that we're pulling back forms from X..]

Morphisms: [Parallel transport] ~~stuff~~
maps are just principal $U(1)$ bundle maps,
transport along interval / homotopy

Composition  Homotopic maps of principal bundles are identified

Get same $\Pi_{\leq 1}$ as just usual Maps$(M, \mathbb{CP}^\infty)$

But now introduce the filtration : $\Pi_{\leq 1}\,\mathrm{filt}_2\,(X_i)^M$
gives $\Pi_{\leq 1}\,\mathrm{Map}(M, \mathbb{CP}^\infty)$, homotopy classes of principal bundle maps get usual topology.

$\boxed{\mathrm{filt}_1 :}$ Objects still same, $U(1)$ bundles + connection
Maps = still principal bundle maps.

P  In filt, can't put arbitrary connection over this 2-simplex to get arbitrary homotopy like we did before...

But curvature form now has at most a 1-form component on Δ — so  both maps must be equal : can't integrate 1-form on 2-simplex...
Maps now not up to homotopy, just usual morphisms of principal bundles : $\Pi_{\leq 1}\,\mathrm{filt}_1$ is correct category of principal bundles with connection...

$\boxed{\mathrm{filt}_0 :}$ Objects principal $U(1)$ bundles + connection.
Maps :  connection on parameter space $M \times$ interval,
but curvature form $\omega \in \Omega^2(M \times [0,1])$ must be only on M part, $\omega \in \Omega^2(M) \otimes \Omega^0([0,1])$
& closed $\Rightarrow$ constant along $[0,1]$

Maps are now horizontal (connection preserving) maps of principal bundles! (connection preserved by this constant family — holonomy of loops can't change due to restriction on curvature ω in family...)

X Eilenberg-MacLane space $\quad X = K(\mathbb{Z}, n)$

$\iota \in Z^n(X, \mathbb{R})$ a generator (assume generates integral cohomology)

$\pi_0 K(\mathbb{Z},n)^M = [M, K(\mathbb{Z},n)] = H^n(M;\mathbb{Z})$

$\pi_0 \text{f.H}_0 (X;\iota)^M$ = Cheeger-Simons cohomology group $\check{H}^{n-1}(M)$ differential characters
= smooth Deligne cohomology $H_D^{n,n}(M)$

$$0 \to H^{n-1}(M;\mathbb{R}/\mathbb{Z}) \to H_D^{n,n}(M) \to [\Omega^n_{cl}, \text{ integer periods}] \to 0$$

- remember closed form representing class & lower dim info.

$\pi_0 \text{ f.H}_n(X,\iota)^M$ - usual $H^n(M)$.

<u>More generally</u> X = space representing some other cohomology theory E, e.g. space of Fredholm operators for K-theory
$\Rightarrow$ differential version of E-cohomology $\check{E}(n)^*(M)$

<u>Differential K-theory - K theory of vector bundles w/ superconnection</u>

<u>Cheeger-Simons</u> $\check{H}^{n-1}(M)$ = homomorphisms $Z_{n-1}(M) \xrightarrow{\chi} \mathbb{R}/\mathbb{Z}$
(character of $n-1$ cycles)
$+ \ \omega \in \Omega^n(M)$ s.t. $\chi(\partial N) = \int_N \omega$

<u>Deligne</u> $\mathbb{Z}(n) = \mathbb{Z} \to \Omega^0 \to \dots \to \Omega^{n-1}$

$$H_D^{m,n}(M) = H^m(M; \mathbb{Z}(n))$$

Cochains: $C(n)^k(M) = (c, h, \omega)$ $\quad c \in C^k(M,\mathbb{Z})$, $\omega \in \Omega^k(M)$
$h \in C^{k-1}(M,\mathbb{R})$

& $\omega = 0$ if $k < n$

$$\delta(c,h,\omega) = (\delta c, h - \omega + c, d\omega)$$

i.e closed if c, ω closed & $\delta h = \omega - c$

m^{th} cohomology group of $C(n)^*(M)$ is $H^{m,n}(M)$.

Third POV see ring structure ~~eg~~. - not theoretical POV

"Ordinary" differential cohomology: $H^{m,n}(M)$ come from "differential functions" into Eilenberg-MacLane spaces

All usual apparatus is present: cup products, integration, pushforward / pullback

G compact Lie group, $\lambda \in Z^4(BG; \mathbb{R})$ "level"

($H^4 = \mathbb{Z}$ for G simple)

M + Principal G-bundle w/ connection $\Rightarrow$ diff function $M \longrightarrow (BG, 2)$ Chern-Weil theory

(we're remembering only forms representing chern classes not connection itself... or even curvature form itself except in $U(1)$ case)

Fix a principal G-bundle P on M $\Rightarrow$ map $\mathcal{A} \longrightarrow \mathrm{Diff}_0(BG, 2)^M$ $\qquad \mathcal{A}$ = space of conns. on P.

Evaluation gives diff fn $\qquad M \times \mathcal{A} \xrightarrow[(c,h,\omega)]{f} (BG, 2)$

dim $M = 2$: $\qquad \int_M (c^* 2, h, \omega)$ integral is a differential function $\mathcal{A} \longrightarrow (\mathbb{C}P^\infty, 2)$

i.e $U(1)$ bundle with connection on moduli space $\mathcal{A}$

$\Rightarrow$ classical Chern-Simons theory (refinement to cocycles of description of Gawedzki)

M. Hopkins II

Space of closed n-forms on M ⟿ as simplicial set, k-simplices $\Omega^n_{cl}(M \times \Delta^k)$

$$\begin{array}{ccc} \Omega^n_{cl}(M \times \Delta^2) & & \\ \downarrow\downarrow\downarrow \partial_i & & \\ \Omega^n_{cl}(M \times [0,1]) & \xrightarrow{\int_{\Delta^k}} & \\ \downarrow\downarrow & & \\ \Omega^n_{cl}(M) & & \end{array}$$

Simplicial abelian gp $\longleftrightarrow$ chain complex

$d = \sum (-1)^i \partial_i$ $\quad$ ∂_i faces

$$\begin{array}{c} \Omega^{n-2}(M) \\ \downarrow d \\ \Omega^{n-1}(M) \\ \downarrow d \\ \Omega^n_{cl}(M) \end{array}$$

Stokes theorem $\Rightarrow$ under $\int_{\Delta^k}$, difference of face maps becomes deRham differential: space of closed n-forms is really the truncated deRham complex of M!

Similarly "space" of n-cocycles on X with values in A $\Longleftrightarrow$ truncated cochain complex $C^0(X;A) \to \cdots \to C^n_{cl}(X;A)$.

Space of differential functions $M \to (K(\mathbb{Z},n), \iota)$

$$\begin{array}{ccc} C(0)^*(M) & \longrightarrow & \Omega^* \\ \downarrow & & \downarrow \\ C^*(M;\mathbb{Z}) & \longrightarrow & C^*(M;\mathbb{R}) \end{array}$$

$C(0)^*(M)$ homotopy pullback of diagram (k-form, integral k-cochain & "path between their images")

$$C(0)^k(M) = C^k(M,\mathbb{Z}) \times C^{k-1}(M,\mathbb{R}) \times \Omega^k$$
$$(c, h, \omega)$$

$$\delta(c,h,\omega) = (\delta c, \omega - c - \delta h, d\omega)$$

Complex of differential cochains:

Cohomologies of all related by long exact Mayer-Vietoris,

-get $H(0)^*(M) = H^*(C(0)^*(M)) \xleftarrow{\simeq} H^*(M;\mathbb{Z})$

Filtration: replace Ω^* by $\Omega^*_{\geq n}$, (s-bcomplex of Ω^*)

$$\begin{array}{ccc} C(n)^*(M) & \longrightarrow & \Omega^*_{\geq n} \\ \downarrow & & \downarrow \\ C^*(M;\mathbb{Z}) & \to & C^*(M;\mathbb{R}) \end{array}$$

Above n nothing changes: $H(n)^k(M) = \begin{cases} H^k(M;\mathbb{Z}) & k>n \\ H^{k-1}(M;\mathbb{R}/\mathbb{Z}) & k<n \end{cases}$

$H(n)^n$: $\quad H^{n-1}(M)\otimes\mathbb{R}/\mathbb{Z} \hookrightarrow H(n)^n(M) \twoheadrightarrow A^n(M)$

$A^n = \{ (x,\omega) \in H^n(M;\mathbb{Z}) \times \Omega^n_{cl} : [\omega] = x \in H^n(M,\mathbb{R}) \}$

$H(n)^n(M)$ = differential characters of Cheeger-Simons

$H(2)^2(M)$ = isom classes of $U(1)$ bundles with connection

Map to A^2: assign curvature + c_1,
kernel is flat line bdls.

Work with category of $U(1)$ bundles with connection $\longleftrightarrow$
work with cochain complex itself, not its cohomology.

Simplicial set of differential fns $M \to (K(\mathbb{Z},k),\iota)$
of filtration n (as simplicial abelian gp)
$\Longleftrightarrow$ truncated chain complex $\cdots \to C(n)^{k-2}(M) \to C(n)^{k-1}(M)$
$\to C(n)^k_{cl}(M)$

- very abelian theory modeled on
E-M space $K(\mathbb{Z},k)$, we'll want less
abelian

V vector bundle over space X.
$\downarrow$
X $\quad \bar{V}$ = Thom complex of V (= 1pt compactification of X cpt)

Classifying space for vector bundles is a manifold, so
can talk about smoothness & transversality in this
context (of maps into X) by classifying them...
Assume X of dim d.

$\mathrm{Map}(M\times\Delta^\bullet, \bar{V})$ simplicial set model for space $\mathrm{Maps}(M,\bar{V})$

$\uparrow$

transverse $\mathrm{Map}(M\times\Delta^\bullet, \bar{V})$ transverse to 0-section

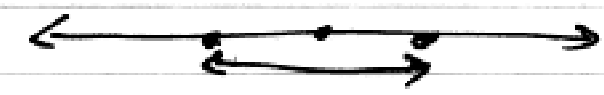
Map is a homotopy equivalence of simplicial sets.

Not nec. true of honest space of maps, but true in simplicial setting...

e.g. $M = pt$ $\quad V = \mathbb{R}$

$\mathrm{Map}(pt, \mathbb{R}^1) = \mathbb{R}^1$, $\quad$ tr map $(pt, \mathbb{R}^1) = \mathbb{R}^1 \setminus 0$.

Simplicially though have paths transverse to 0, so do capture right topology!



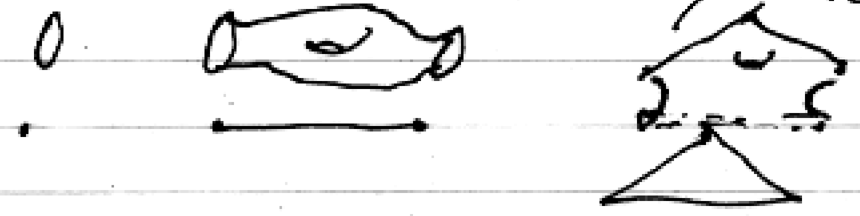
(Thom)

Transverse map $f \xrightarrow{f^{-1}(0)}$ submanifold $\Sigma \subset M \times \Delta^k$

+ map $\Sigma \longrightarrow X$ classifying the normal bundle of Σ.

Think of X as some classifying space, eg for spin structures...

Get Manifolds with boundary corners etc



$$\begin{tikzcd} \Sigma \arrow[r] \arrow[dr] & M \times \Delta^k \arrow[d] \\ & \Delta^k \end{tikzcd}$$

Get simplicial set. $\quad$ 0-simplices = manifolds in X

1-~~simplices~~ simplices are cobordisms in X

2-simplices cobordisms between cobordisms...

— Contains same homotopy information as the function space again...

<u>Differential functions as topological field theories</u>

$(X, z) \qquad z \in Z^d(X, \mathbb{Z})$

$M \xrightarrow{(c,h,\omega)} (X, z) \qquad \Rightarrow (c^*z, h, \omega) \in \check{Z}(d)^d(M)$

X takes care of integrality: forces $c^*\iota$ to be integral, in fact pulled back from X, might store complicated relations among cohomology classes.

M $d-1$ manifold $\Rightarrow$

$$M^{d-1} \times S \longrightarrow (X, z) \quad \text{gives}$$

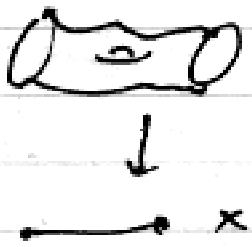
$$Z(d)^d(M^{d-1}\times S) \Rightarrow \int_M (c^* z, \int \omega) \in Z(1)^1(S)$$

$\Rightarrow$ class in $H(1)^1(S) = \mathrm{Maps}(S, U(1))$

class is $e^{2\pi i \int_M h}$, think of as topological term in a Lagrangian

$$M^{d-2} \times S \longrightarrow (X, z) \Rightarrow Z(d)^d(M^{d-2}\times S) \xrightarrow{\int_M} Z(2)^2(S)$$

$= U(1)$ bundle + connection



$\times S \rightarrow (X, z) \Rightarrow Z(2)^2(S\times\Delta^1)$

$\times S$

i.e isomorphism of principal bundles

Hermitian line associated to any $d-2$ manifold & isomorphisms between these lines for any cobordism

- Can get interesting relations among cohomology classes captured by X

Example $X = G$ $\quad z \in Z^3(X) \Rightarrow$ WZW term

$X = BG$ $\quad z \in Z^4(X) \Rightarrow$ Chern-Simons term

$M \rightarrow BG \Longleftrightarrow$ principal G-bundle on M

Can refine into differential function using a connection on bundle — i.e differential fns into spaces classify geometric data.

A Geometric Story

M = Spin^c manifold, $\not{D}$ Spin^c Dirac operator.

(tangential char. class)

Degree 4 component : $(\text{index } \not{D})_4 = \frac{c^2}{8} - \frac{p_1}{24} =: \kappa(c)$

$c = c_1$ of Spin^c bundle. This is not an integer cochain even though gives integers on spin^c 4-manifold
— not encoded in differential forms / ordinary cohomology, will need to pass to other cohomology theories that feel this integrality.

Variation of Spin^c structure : change by a Hermitian line bundle, will change $c \mapsto c - 2x$
x line bundle or its c_1.

3d TFT

$$\kappa(c-2x) - \kappa(c) = \frac{x^2 - xc}{2}$$

$\Rightarrow$ field theory on space of Spin^c structures ...
input = 2-dim cohomology theory, on 3-mfld enters in Lagrangian, 2-mfld will define a line bundle on space of x's.
even though not integral ... def of $\not{D}$.

Degree 6 component . $(\text{index } \not{D})_6 = \frac{c^3 - p_1 \cdot c}{48} = \kappa(c)$

$\Rightarrow$ 5d TFT, with fields 2-form / spin^c structures x

$$\kappa(c+2x) - \kappa(c) = \frac{x^3}{6} + \frac{cx^2}{2} + \frac{(3c^2 - p_1)}{24} x$$
$$= \frac{x^3}{6} + \dots$$

For M-theory 5-brane encounter $\frac{x^2 - x\lambda}{2}$
where x is a 4-form (differential 4-cocycle)
- no obvious index theory interpretation, must couple to 4-form.

M-theory action $\frac{x^3}{6} + \dots$ on 11-manifold

Diaconescu - Freed - Moore, Witten describe these field theories via E_8 index theory

We'll describe a different more topological approach.

"Nonabelian" nature of these theories!

ex 3-d TFT assoc. to Pfaffian of $\not{D}$

$M^3 \rightsquigarrow$ point in $U(1)$
$M^2 \rightsquigarrow$ Hermitian line
cobordism $\rightsquigarrow$ unitary map of Hermitian lines

disjoint union of 2-mflds $\Rightarrow$ $\otimes$ of Hermitian lines
" " 3- " $\Rightarrow$ product in $U(1)$

get Symmetric monoidal structure, in fact Picard category:
groupoid with symm. monoidal $\otimes$ s.t. every object invertible

$\mathcal{C}$ Picard category $\Rightarrow$ $A = \pi_0 \mathcal{C}$ set of Isom classes
$B = \pi_1 \mathcal{C}$ = automorphism group $\mathrm{aut}(e)$ e = identity object

Automorphisms of any other object a is canonically $\cong \mathrm{aut}(e)$
– tensor with identity map of a^{-1}.

[e.g $\mathcal{C}$ = fund groupoid of space with homotopy commutative group structure.]

$a \in \mathcal{C}$ $\Rightarrow$ flip $a \otimes a \xrightarrow{\text{flip}} a \otimes a$ $\Rightarrow$ elmt of $B = \mathrm{aut}(e)$
$\Rightarrow$ map $\underline{A \otimes \mathbb{Z}/2 \longrightarrow B}$ $\underline{\text{k-invariant}}$ of $\mathcal{C}$

for $n \geq 2$ $[K(A,n), K(B,n+2)] = \mathrm{Hom}(A \otimes \mathbb{Z}/2, B)$
(Eilenberg-MacLane) $= H^{n+2}(K(A,n); B)$

Homotopy fiber
$$\begin{array}{ccc} X & \longrightarrow & K(A,n) \\ & & \downarrow \\ & & K(B,n+2) \end{array} \implies \text{k-invariant}$$

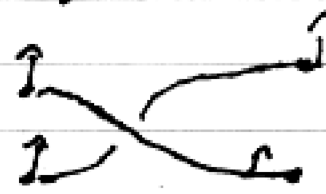
$\pi_n(X) = A$
$\pi_{n+1}(X) = B$

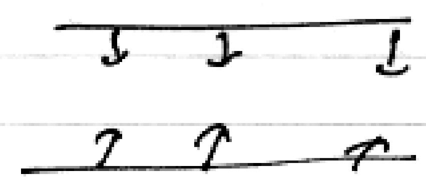
Picard categories $\Longleftrightarrow$ spaces with only 2 consecutive homotopy gps_ only data is k invariant

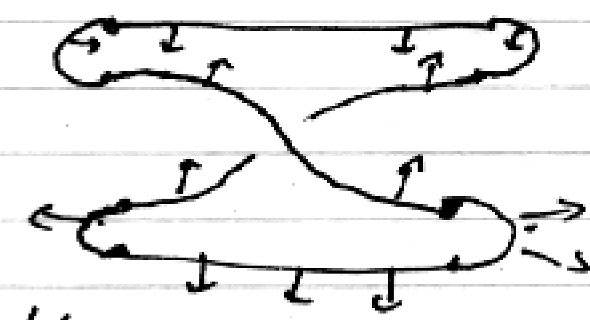
e.g $\mathcal{C}$ = Hermitian lines : $\pi_0 \mathcal{C}$ trivial, $\pi_1 \mathcal{C} = U(1)$

Suppose Σ RS with non-bounding spin structure — mod 2 index of $\not{D}$ not 0.

draw as $\updownarrow$ spin up

Compute k-invariant  perform spin cobordism

compare with opposite that cobordism 

$\Rightarrow$ composite cobordism 

$\Rightarrow$ 3-manifold, calculate mod 2 index $e^{\pi i \eta(\Sigma \times S^1)} = -1$ η-invariant ....

$\Rightarrow$ nontrivial k-invariant for ~~the~~ this theory — so not quite cohomological, mixes Eilenberg-MacLane spaces ... nonabelian.

— We're trying to calculate a monoidal functor from the spin cobordism category to Hermitian lines but get contradiction! Hermitian lines have 0 k-invariant, spin cobordism category doesn't, so can't map these two Picard categories — so need to go to graded Hermitian lines for our field theory.

[k-invariant comes Steenrod operation]