M. Hopkins III

Spin^c Dirac operator $\Longrightarrow$ 3D & 5D TFIs

3D: $\int_M \frac{c^2}{8} - \frac{p_1}{24} \in \mathbb{Z}$ on Spin^c 4-manifold

5D $\int_M \frac{c^3 - p_1 c}{48} \in \mathbb{Z}$ on Spin^c 6-manifold

Can generalize in 2 directions: look at higher terms, eg
7D theory $\frac{c^4}{2^4 4!} + \ldots$ etc coming out of Dirac operator
on 8-manifolds ... or can generalize to higher dim cohomology classes instead of c, generalize Spin^c structures

Need to calculate stable homotopy groups of Eilenberg-MacLane spaces - maximally bad, if anything can go wrong it will

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Spectra $\quad E: \{ E_n, \quad E_n \xrightarrow[f_n]{\sim} \Omega E_{n+1} \quad n \in \mathbb{Z} \}$

homotopy eq, or better homeomorphism...

E_n give cohomology theory $E^n(X) = [X, E_n]$

Cobordism spectra: $\quad MO(n) = \text{Thom}(BO(n); \xi_n)$ universal bundle

$\Sigma MO(n) \longrightarrow MO(n+1) \iff MO(n) \longrightarrow \Omega MO(n+1)$

$\Rightarrow$ Spectrum MO with $MO_n = \lim_{N \to \infty} \Omega^N MO(N+n)$

Suppose $B \longrightarrow BO$ some general classifying space, fibration / BO

$\Rightarrow$ define

$$\begin{array}{ccc} B(n) & \longrightarrow & B \\ \downarrow \chi_n & & \downarrow \\ BO(n) & \longrightarrow & BO \end{array}$$

, have spectrum $B^\chi = \{ B^\chi_n = \lim_{N \to \infty} \Omega^{N+n} B(n+N)^{\chi_n} \}$

$B(n)^{\chi_n}$ = Thom space of bundle classified by map χ_n.

S smooth manifold, compact $\Rightarrow$ factor maps into MO_n

$$S \longrightarrow MO_n \quad \Longleftrightarrow \quad \mathbb{R}^N \times S \longrightarrow MO(n+N)$$
$$\Omega^N MO(n+N)$$

$\Longrightarrow$ $E \subset \mathbb{R}^N \times S$.
manifold of codim $N+n$

$\Longleftrightarrow$ $\begin{matrix} E \\ \downarrow \\ S \end{matrix}$ rel dim $-n$. So maps to $MO(n)$ classify manifolds + embedding into a big Euclidean space, in limit get just the cobordism groups . . . .

Differential cohomology groups:

E spectrum $\{E_n,\ \sigma_n : S^1 \times E_n \longrightarrow E_{n+1}\}$

$\rightarrow$ maps on cochains $C^*(E_{n+1}) \longrightarrow C^*(S^1 \times E_n) \xrightarrow{/[S^1]} C^{*-1}(E_n)$

$\rightarrow$ make an inverse system, & define $C^*(E) = \varprojlim C^{*+n}(E_n)$
$[* \in \mathbb{Z}]$

Cocycle $z \in Z^k(E) \Longleftrightarrow \{z_n \in Z^{n+k}(E_n)\}$ compatible family

$\pi_k E := \pi_{n+k} E_n \qquad k \in \mathbb{Z}$

Basic fact: $H^*(E;\mathbb{R}) = \mathrm{Hom}(\pi_* E, \mathbb{R})$
... even rationally... homotopy & homology very close here...

Write $V := \pi_* E \otimes \mathbb{R}$, take cohomology with coeffs in V
(graded abelian group)
$\longrightarrow$ $\exists$ canonical cohomology class, "fundamental coh. class")
in $H^0(E;V) = \mathrm{Hom}(\pi_* E, V) = \mathrm{Hom}(\pi_* E \otimes \mathbb{R}, \pi_* E \otimes \mathbb{R})$
$\ni \mathrm{id}$

<u>Choose</u> cocycle $z \in Z^0(E;V)$ representing the fundamental cohomology class.

<u>Def</u> $E(n)^k(M) = \pi_N \,\mathrm{filt}_{k+N-n}(E_{N+k}, z)^M$ differential E-cohomology

M smooth manifold

$$E(n)^k(M) = \begin{cases} E^k(M) & k \geq n \\ E^{k-1}(M;\mathbb{R}/\mathbb{Z}) & k < n \end{cases}$$

$$E^{n-1}(M)\otimes\mathbb{R}/\mathbb{Z} \hookrightarrow E(n)^n(M) \twoheadrightarrow A^n_E(M)$$

where
$$\begin{array}{ccc} A^n_E(M) & \longrightarrow & \Omega_{cl}(M;V)^n \\ \downarrow & & \downarrow \\ E^n(M) & \xrightarrow[\text{use } 2]{} & \cancel{H^n(M;V)} \\ & & H^n(M;V) \end{array}$$

closed forms on M with periods controlled by integrality properties in V.

Anderson Duality : "Grothendieck duality" for spectra

$\pi_4 MSpin^c$ = cobordism gp of 4-dim $Spin^c$ manifolds

$$\begin{array}{c} \pi_4 MSpin^c \\ \downarrow \int \frac{c^2}{8} - \frac{p_1}{24} \\ \mathbb{Z} \end{array}$$

Want to replace such a homomorphism on (homotopy) grps by a map of spectra...

$\Rightarrow$ Spectrum $\tilde{I}$ with property

$$\mathrm{Ext}(\pi_{k-1}E, \mathbb{Z}) \hookrightarrow [E, \Sigma^k \tilde{I}] \twoheadrightarrow \mathrm{Hom}(\pi_k E, \mathbb{Z})$$

"universal coeff sequence"

Can define $I_{\mathbb{Q}}$ representing $\mathrm{Hom}(\pi_*^{st}(-), \mathbb{Q}) = [-, I_{\mathbb{Q}}]$

similarly $I_{\mathbb{Q}/\mathbb{Z}}$ $\qquad \mathrm{Hom}($ " $\quad , \mathbb{Q}/\mathbb{Z}) = [-, I_{\mathbb{Q}/\mathbb{Z}}]$

& then define $\tilde{I}$ via $\tilde{I} \to I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}}$ homotopy fiber.

Pontrjagin duals to homotopy grps of spheres.

k	$\pi_k \tilde{I}$
>0	0
0	$\mathbb{Z}$
-1	0
-2	$\mathbb{Z}/2$
-3	$\mathbb{Z}/2$
-4	$\mathbb{Z}/24$

$\tilde{I}(1)^1(M) = \text{Smooth}(M, U(1))$ since first gps of $\hat{I}$ look like $K(\mathbb{Z}, 1)$

$\tilde{I}(2)^2(M)$ = iso classes of graded $U(1)$-bundles + connection
– groups $\mathbb{Z}, \mathbb{Z}/2$ with nontrivial k-invariant

$\tilde{I}(3)^3(M)$ – graded gerbes + conn / bundles of central simple algebras, graded
– gerbe + discrete pieces of information

$\tilde{I}_2 = \mathbb{Z}_2 \times \mathbb{CP}^\infty$.

$\tilde{I}(3)^3$: "graded Brauer group"

General Story $M\langle G\rangle$: some cobordism theory, classifying something. "G-structures" eg. Spin-cobordism

Search for integer invariant of d-dimensional "G"-manifolds
$\Longleftrightarrow \pi_d M\langle G\rangle \longrightarrow \mathbb{Z}$, & refine into a map $M\langle G\rangle \longrightarrow \Sigma^d \hat{I}$.
[if $\pi_{d-1} M\langle G\rangle = 0$ then this comes for free]

$M^{d-1} \downarrow S$ family of $d-1$ manifolds, + e.g. metric on fibers, ... extra grading

$\Rightarrow$ diff function

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & (M\langle G\rangle, 2) \arrow[d] \\ & (\tilde{I}, 2), \end{tikzcd}$$

$\Leftrightarrow$ smooth function $S \longrightarrow U(1)$

$M^{d-2} \Longrightarrow$ graded $U(1)$ bundle + connection on S .....
– source of TFTs from integer valued cobordism invariants...

M^4 Spin^c, $\quad K(c) = \frac{c^2}{8} - \frac{p_1}{24} = \frac{c^2 - L_4}{8}$ $\quad$ Hirzebruch L-poly

$$q(x) = K(c+2x) - K(c) = \frac{x^2 - xc}{2}$$

$q(x+y) - q(x) - q(y) = xy$: q quadratic refinement of the intersection pairing...

Suppose L lattice + bilinear form $\langle , \rangle$ $\quad [L = H^{p}(M;\mathbb{Z})/\text{torsion}]$

What do we need for quadratic refinement of $\langle , \rangle$?

$L \xrightarrow[\langle x,x \rangle]{} \mathbb{Z}/2$ $\quad$ nondegenerate $\Rightarrow \langle x,x \rangle = \langle x, \bar{c} \rangle$

$\bar{c} \in L \otimes \mathbb{Z}/2$

Characteristic element $c \in L$ choice of element with $c = \bar{c} \bmod 2$

$c^2 - \text{sign}(L) \equiv 0 \pmod 8$

$\updownarrow$

quadratic refinement : $q = \frac{x^2 - xc}{2}$

M^{4k} oriented $\Rightarrow$ char element for $2k$ cohomology is v_{2k}, $2k^{\text{th}}$ Wu-class, $x^2 = x v_{2k} \in H^{4k}(M;\mathbb{Z}/2)$

Equip M with an integer lift of v_{2k}

$$\begin{tikzcd}
 & BSO\langle v_{2k} \rangle \arrow[r] \arrow[d] & K(\mathbb{Z}, 2k) \arrow[d] \\
M \arrow[r] \arrow[ur, dotted] & BSO \arrow[r, "v_{2k}"] & K(\mathbb{Z}/2, 2k)
\end{tikzcd}$$

$k=1$: $BSO\langle v_2 \rangle = B\mathrm{Spin}^c$!

So Wu structure generalizes Spin^c.

Lift (1) $\pi_{4k} MSO\langle v_{2k} \rangle \to \mathbb{Z}$

to (2) $MSO\langle v_{2k} \rangle \to \Sigma^{4k} \tilde{I} \Rightarrow$ TFT.

Extra data needed for the lift (2) was defined by Milgram, Morgan-Sullivan

On Spin our Wu class is Stiefel Whitney w_4

$$\begin{tikzcd} BSpin \times K(\mathbb{Z},4) \arrow[r] \arrow[d] & K(\mathbb{Z},4) \arrow[d] & \ni c_2 \text{ Chern} \\ BSpin \arrow[r, "w_4"] & K(\mathbb{Z}/2,4) & \end{tikzcd}$$

But $\pi_7 \, MSpin \wedge K(\mathbb{Z},4)_+ = 0 \Rightarrow$ no extra data needed for lift to $\tilde{I}$.

(w/Freed) : Cubic analog of this story

For M-theory action need. $MSpin \wedge K(\mathbb{Z},4)_+$

Spin-manifolds + integer lift of 4th Wu class

Story : $\pi_{11} \, MSpin \wedge K(\mathbb{Z},4)_+ = 0$ ∴ so no extra info for refinement.

Hopkins IV

Hirzebruch genera: ring homs $\binom{\text{cobordism}}{\text{ring}} \xrightarrow{\varphi} R$

Cobordism theories: MU - stably almost complex mflds $\leftrightarrow$ "complex cobordism"
MSO - oriented cobordism
$MSpin$ - spin cobordism
$MString$ - "string cobordism" $(= MO\langle 8\rangle)$

Stable normal bundle $\leftrightarrow$ $M \to BO$, orientation: lift to $\begin{array}{c} BSO \\ \downarrow \\ BO \end{array}$
universal cover

$$\begin{array}{ccccccc} BString & \to & BSpin & \to & BSO & \to & BO \\ \text{[3-conn]} & & [\text{2-conn}]\ \text{kill } \pi_2 & & [\text{simply conn}] & & \\ \lambda = \frac{p_1}{2} = 0 & & w_2 = 0 & & w_1 = 0 & & \end{array}$$

$MU_* \otimes \mathbb{Q} = \mathbb{Q}[[\mathbb{CP}^1], [\mathbb{CP}^2], \ldots]$
$MSO_* \otimes \mathbb{Q} = \mathbb{Q}[[\mathbb{CP}^2], [\mathbb{CP}^4], \ldots]$

Cobordism: structure is all on (stable) <u>normal</u> bundle to manifold!

If R is torsion free, φ genus is determined by its values $\varphi([\mathbb{CP}^n])$

$\log_\varphi(x) := \sum \varphi(\mathbb{CP}^n) \frac{x^{n+1}}{n+1}$

$\exp_\varphi(x) = \log_\varphi^{-1}(x)$

characteristic series $K_\varphi(x) = \frac{x}{\exp_\varphi(x)}$

Hirzebruch: $\varphi(M) = \int_M \prod K_\varphi(x_i) \quad \in R \otimes \mathbb{Q}$

x_i: roots of TM - formally $TM = L_1 + \ldots + L_{d/2}$ complex line bundles
$x_i = c_1(L_i)$

Theorem (Quillen): $\phi(M) \in R \quad \forall M$ iff power series
$F(x,y) = \exp_\varphi(\log_\varphi(x) + \log_\varphi(y)) \in R[[x,y]] \subset R \otimes \mathbb{Q}[[x,y]]$?

Todd genus: $K = \frac{x}{1-e^{-x}}$ $\quad F(x,y) = 1-(1-x)(1-y) \in \mathbb{Z}[[x,y]]$

Lift genera to maps of spectra:
instead of $MU_* \longrightarrow R$ look for multiplicative map of cohomology theories $MU \longrightarrow E$, $R = \pi_* E = E(pt)$

$MU(pt) \leftrightarrow$ individual manifold, $MU(S) \leftrightarrow$ families of manifolds & cobordism. So Hirzebruch genera are invariants of individual manifolds, while $MU \to R$ gives invariants of families...
e.g. Todd genus $MU \longrightarrow K$
Grothendieck: generalize RR (Todd) to families $\Rightarrow$ K-theory ... same source.

eg $MO^{-n}(S) \longleftrightarrow S \longrightarrow \lim_{N\to\infty} \Omega^{N+n} MO(N)$

S compact $\Rightarrow$ lands in some finite stage, $R^{N+n} \times S \longrightarrow MO(N)$

$$\begin{array}{ccc} R^{N+n}\times S & \longrightarrow & MO(N) \\ \uparrow & & \uparrow \text{ 0-section} \\ E & \longrightarrow & BO(N) \end{array}$$

E manifold living in $R^{big} \times S$
$\Rightarrow$ family of manifolds of rel dim n....

'80s Ochanine: genera given by elliptic integrals, $\log_\varphi = \int \frac{dx}{\sqrt{1-2\delta x^2+\varepsilon x^4}}$
- any genus multiplicative in ~~certain types~~ fibrations with connected structure group is a specialization of this.

$F(x,y)$ = group law of elliptic curve $y^2 = 1 - 2\delta x^2 + \varepsilon x^4$.

Witten's variation - Witten genus $K_{\varphi_W} = \frac{x}{\sigma(x)}$ σ = Weierstrass sigma
(supercharge of nonlinear σ-model)
- signature of $\mathcal{L}M \Leftarrow$ Ochanine
index of Dirac on $\mathcal{L}M \Leftarrow$ Witten

$\varphi_W(M^{2n})$ = modular form of wt n

~~Idea~~ Question: "families version" ① $MSO \xrightarrow{\varphi_{Ochanine}} E$
- topological version ② $MString \xrightarrow{\varphi_{Witten}} E'$

① Landweber-Ravenel-Stong, $E^*(X) = MSO^*(X) \otimes_{MSO^*(pt)} R$
where $R = \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon, \Delta^{-1}]$
- elliptic cohomology $Ell^*(X)$
- need exactness for this tensor product for it to define a cohomology theory, ie satisfy Mayer-Vietoris

Landweber exactness also holds for Todd genus, could define $\mathbb{C}$ K-theory this way from MU.
Hopkins-Hovey: same for KO using MSp...

What is $E^{\cdot}$ and what is geometric meaning of Ell^*?

Problems: • we've inverted 6, • we've picked Jacobi quartic & ignored all symmetries of the curve.

2 ⇒ tmf, topological modular forms

return to K theory: $F(x,y) = 1-(1-x)(1-y)$ $\quad \mathbb{G}_m$
One automorphism $(1-x) \longmapsto (1-x)^{-1}$,
gives symmetry of K-theory, namely complex conjugation.
⇒ KR Real K-theory: if X has trivial involution
fixed points $KR(X) = KO(X)$, takes into account symmetry of $\mathbb{G}_m$.

To construct tmf:
1. make a functor $\left(\substack{\text{cat. of} \\ \text{elliptic curves}}\right) \xrightarrow{E}$ Spectra
on the nose, not just up to homotopy: need to rigidify homotopy gp action, obstructions to this (homotopy Moore space problem)
Also different ways to rigidify which are very different eg $\mathbb{Z}/2 \circlearrowright S^1$ up to homotopy is trivial.

— analog of KR, functor from category of multiplicative gps to spectra.

2. $\mathrm{tmf} = \lim\limits_{\text{ell curves } C} E(C)$ $\quad$ [Hopkins-Miller]

Deformation complex for stack of elliptic curves over stack of formal groups is supported away from supersingular locus, where mostly automorphisms are just $\mathbb{Z}/2$, obstructions understood there ... like KR ... then pray!

Topological index of $\not{D}$ on $\mathcal{L}M$ (Ando-Hopkins-Rezk)
— genus $MString \longrightarrow \mathrm{tmf}$

Relating K theory to KO: spectral sequence

$H^*(\mathbb{Z}/2; K^*(X)) \Rightarrow KO^*(X)$

X = pt: $H^*(\mathbb{Z}/2; K^*) \Rightarrow KO^*(pt)$

For tmf: $H^s(\mathcal{M}, \omega^{\otimes t}) \Rightarrow \pi_{2t-s}\, tmf = tmf^0(S^{2t-s})$

use to calculate homotopy gps of spheres! $\mathcal{M}$ = moduli of curves

- think of as $H^s(SL_2\mathbb{Z}, Hol(\text{upper half plane}))$ - complex vars

$H^0(\mathcal{M}, \omega^t)$ = modular forms of weight t / $\mathbb{Z}$

Invert 6 - get rid of all higher cohomology of stack, orbifold pts order 2, 3

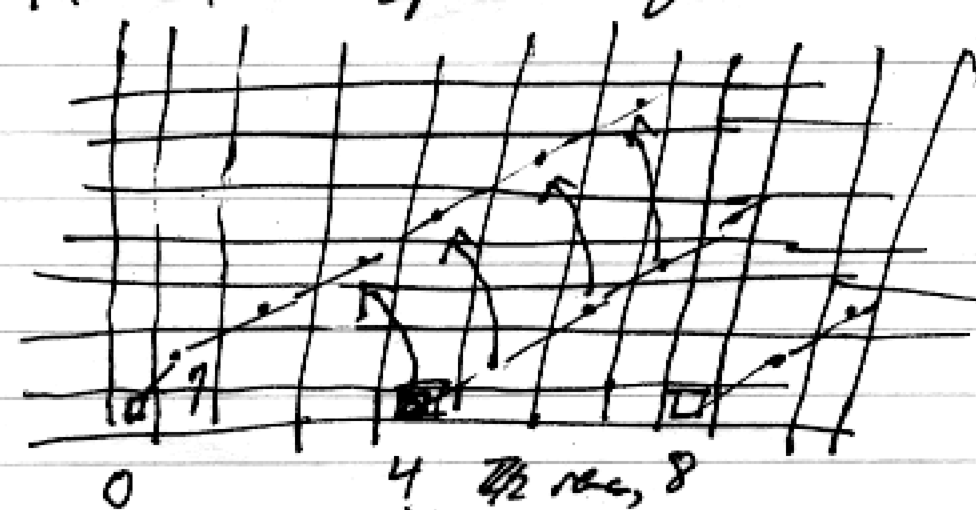
Draw spectral sequence: $H^s(-, \deg t)$ Serre: $-s$, $-t$

diagonals all contribute to one group

topological indexing s, $t-s$

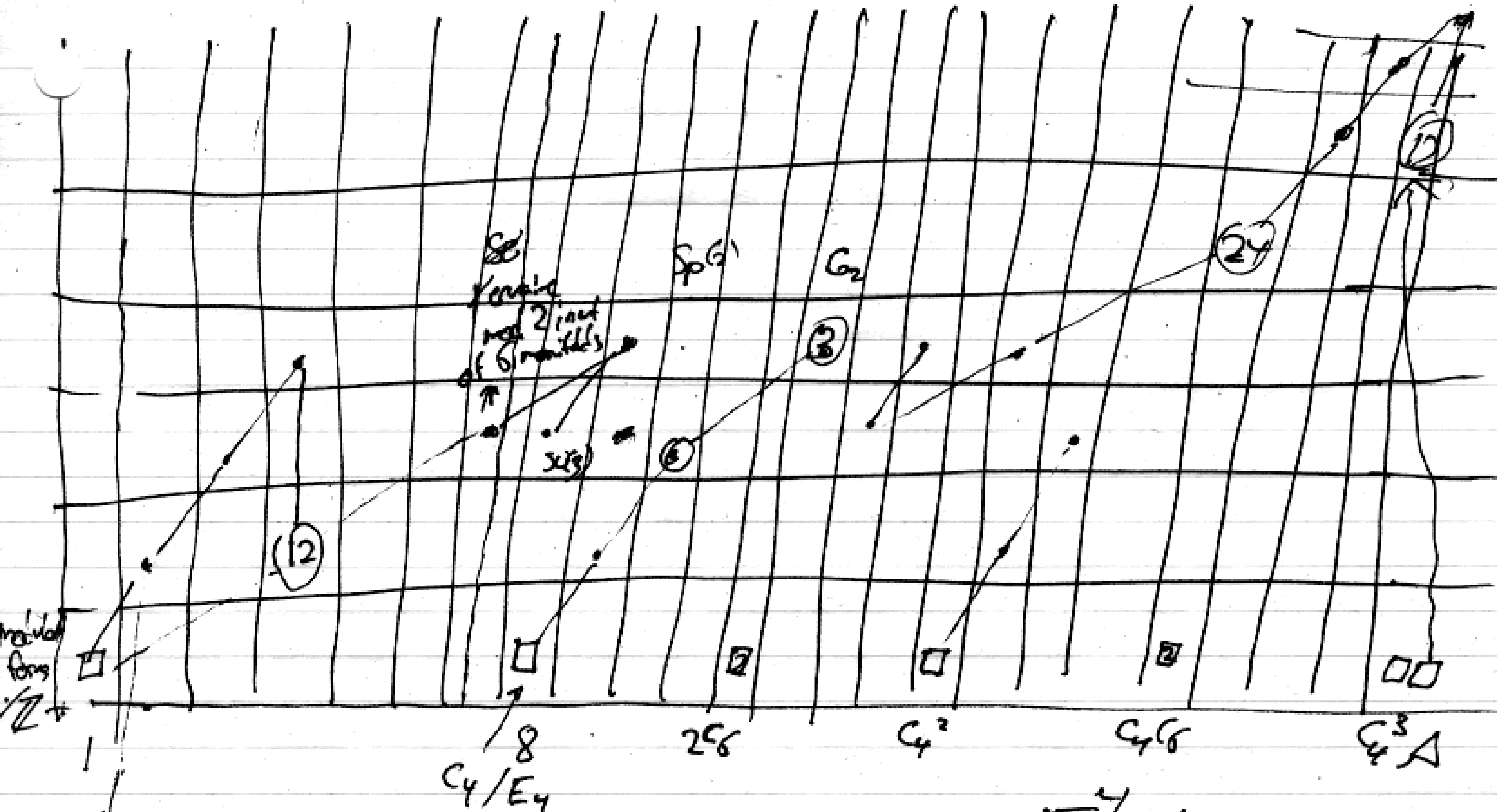
all ~~diffs~~ things contributing to a given group lie in a line

K → KO spectral sequence: $\square = \mathbb{Z}$ $\bullet = \mathbb{Z}/2$



lines: connected by multiplication by η

$\boxed{2}$ factor of 2 comes in $\hat{A}$ genus at



E_4^4-term

$\bullet = \mathbb{Z}/2$ $\quad$ (n) $\cdot\ \mathbb{Z}/n$

| : nontrivial gp extension, $\mathbb{Z}/24$

(12) $\quad \mathbb{Z}/24 \quad \longleftrightarrow \quad \eta$-invariant on 3-manifold
mod 24 index of $\not{D}$ on loop space of 3-manifold

In dim 8 have new mod 2 invariant, nontrivial on $SU(3)$

Can test first several invariants nontrivially on Lie groups.

Differential in dimension 24 $\Longrightarrow$ $\exists$ 23 sphere with nontrivial differentiable structure, bounds a 24 dim string manifold . . .

Find out extra divisibilities of genera from differentials eg Rokhlin's theorem.

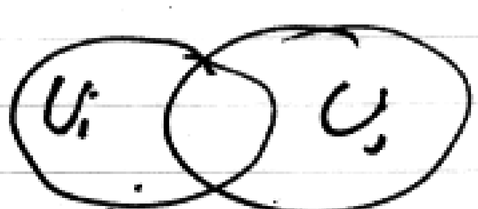
M^{24} string manifold $\Longrightarrow$ Rarita-Schwinger $[M] =$
$\hat{A}(M, T_{\mathbb{C}}) \equiv 0 \pmod{24}$

— e.g. on 23-manifold determinant of this RS operator has a 24^{th} root! — consequences for TFT

M. Hopkins V

Twisted K-Theory (w/ Freed, Teleman; V. Braun, C. Douglas, S. Schäfer-Nameki)

Defining vector bundles by gluing:
 - isomorphisms on intersections between vector bundles on opens



$V_i \underset{\lambda_{ij}}{\cong} V_j$

$\boxed{\lambda_{jk}\lambda_{ij} = \lambda_{ik}}$ cocycle eqn

Twisting of K-theory: on $U_i \cap U_j$ have L_{ij}, "automorphism of K-theory on $U_i \cap U_j$"
$\rightsquigarrow$ twisted vector bundle: V_i vector bundles on U_i,
$\lambda_{ij} : L_{ij} V_i \longrightarrow V_j$ on overlaps

To make sense of cocycle equation, need $L_{jk} \circ L_{ij} = L_{ik}$
$\{L_{ij}\} \longleftrightarrow$ a twisting τ of K-theory, $\Rightarrow$ $K^{\tau}(X)$

Get examples of twistings from $\pm$ a line bundle - i.e. a graded line bundle L_{ij} on $U_i \cap U_j$ acting by tensoring
$\rightarrow$ 1-cocycle valued in graded line bundles
$\tau = \{L_{ij}\} \in H^1(X, \mathbb{Z}/2) \times H^3(X, \mathbb{Z})$

Variations: • twist K-homology • twist equivariant K-theory

e.g. $X = S^3$, $\tau = k \in H^3(S^3; \mathbb{Z}) = \mathbb{Z}$.

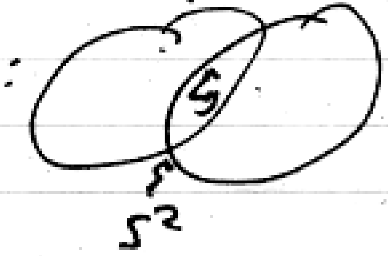
write $S^3 = U_1 \cup U_2$ U_i = complement of i^{th} pole,
$U_1 \cap U_2 \sim S^2$.

Mayer-Vietoris $0 \to K^{\tau+0}(S^3) \to K^{\tau+0}(U_1) \oplus K^{\tau+0}(U_2) \to K^{\tau+0}(U_1 \cap U_2)$
$\hookrightarrow K^{\tau+1}(S^3) \longrightarrow \cdots$

$\tau|_{U_i} = 0 \Rightarrow$ $0 \to K^{\tau+0}(S^3) \to \mathbb{Z} \oplus \mathbb{Z} \longrightarrow K(S^2) = \mathbb{Z} \oplus \mathbb{Z}$
$\to K^{\tau+1}(S^3) \to 0$

Twisting k $\rightsquigarrow$ k^{th} power of Hopf line bundle L on $S^2 \sim U_1 \cap U_2$

S^3:



$$K(U_1) \oplus K(U_2) \longrightarrow K(U_1 \cap U_2)$$

$\mathbb{Z}$ (trivial line) $\oplus$ $\mathbb{Z}$ (trivial line) $\qquad \mathbb{Z}$ (1 trivial) $\oplus \mathbb{Z}$ ($x = (L-1)$ reduced Hopf bundle L)

Write map in a matrix:

$$\begin{tikzcd} 1 \arrow[rr, bend left] \arrow[rrr, bend right] & 1 & \longrightarrow \; 1 & x \end{tikzcd} \qquad L^k = 1 + kx$$

$\begin{pmatrix} 1 & 1 \\ 0 & k \end{pmatrix}$ monomorphism, cokernel cyclic order k:

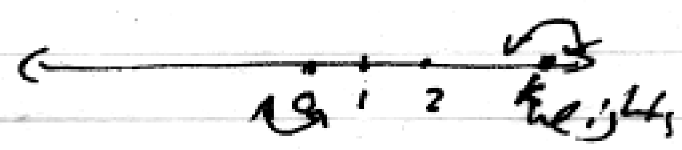
$$K^{\tau+0}(S^3) = 0 \qquad K^{\tau+1}(S^3) = \mathbb{Z}/k.$$

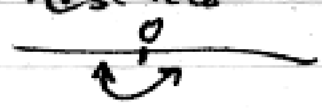
$S^3 = SU(2)$

$K^{\tau+*}_{SU(2)}(SU(2))$: (acting by conjugation) $K^0_{SU(2)}(pt) = R[SU(2)]$, $K^0_{SU(2)}(SU(2)/T) = R[T]$

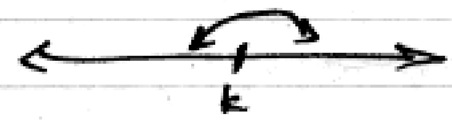
$$\Rightarrow \quad R[SU(2)] \oplus R[SU(2)] \longrightarrow R[T]$$

One copy of $R[SU2]$ just get restricted, i.e. get invariants of flip





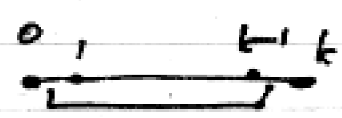
Other copy gets shifted, get invariants of flip



Kernel: invariant under both flips: $(a,b) \mapsto 0$

a invariant so can't have finite linear combo of wts, so kernel $= 0$

cokernel is fundamental domain - free abelian group on weights $1, \dots, k-1$



Theorem (F-H-T) $\quad K^{\tau + \dim G}_G(G) = Rep^{\tau - \sigma}(LG)$ loop group

reps of level $\tau - \sigma$, σ level of adjoint

$K^G_\tau(G)$ K-homology ring is Verlinde algebra at level $\tau - \sigma$

Covering by more than 2 elements $\Rightarrow$ Mayer-Vietoris spectral sequence.
- calculate for all Lie groups

More general coverings

$$\begin{array}{ccc} U_{ij} = U_i \times_X U_j & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ U_j & \longrightarrow & X \end{array}$$

Replace open covers by more general

- we'll use the path space of X and assume path connected

$PX \longrightarrow X$.

$$PX \times_X PX \simeq PX \times \Omega X$$

get a path & a loop - up to homotopy

loops & a point on the loop.

eg $X \sim BG$ $\quad PX \sim EG$, $\quad PX \times_X PX \sim EG \times G$.

for X a group: $PX \times_X PX$ is actually homeomorphic to $PX \times \Omega X$

$PX \times_X \cdots \times_X PX \sim PX \times \Omega X \times \cdots \times \Omega X$ $\quad$ natural higher order self intersections for every order

Mayer-Vietoris spectral sequence: $\quad$ G. Segal: Classifying spaces & spectral sequences IHES 1968 www.numdam.org]

If $K_*(\Omega X \times \cdots \times \Omega X)$ has a perfect Künneth formula

$= K_*(\Omega X) \otimes \cdots \otimes K_*(\Omega X)$ $\quad$ [automatic if $H_{odd}\, \Omega X = 0$ eg $X = G$ compact Lie grp]

$\Rightarrow$ E_2-term $\quad \mathrm{Tor}^{K_*(\Omega X)}(K_*(pt), K_*(pt)) \Longrightarrow K_*^{\tau}(X)$

Spectral sequence here works wonderfully well!

Example $X = SU(n)$. $\quad \mathbb{C}P^{n-1} \longrightarrow \Omega U(n)$

line $\rightarrow$ loop of rotating $e^{2\pi i t}$ on this line, fixing orthogonal complement

Can translate this loops fact into $\Omega SU(n)$

- this embedding controls topology of $\Omega SU(n)$:

$K_* \Omega SU(n) = \mathrm{Sym}\, K_* \mathbb{C}P^{n-1} / b_0 = 1$

$K_* \mathbb{C}P^{n-1} = \mathrm{Hom}(K^*(\mathbb{C}P^{n-1}), \mathbb{Z})$ = free abelian group on $b_0, b_1, \ldots b_{n-1}$

since $K^*(\mathbb{C}P^{n-1}) = \mathbb{Z}[x]/x^n$ $\qquad x = (L-1)$

$K_* \Omega SU(n) = \mathbb{Z}[b_1 \ldots b_{n-1}]$

Untwisted $K_*(SU(n))$: $\mathrm{Tor}^{K_*(\Omega SU_n)}(\mathbb{Z},\mathbb{Z})$

under map $K_*(\Omega SU(n)) \to \mathbb{Z}$, $b_i \mapsto 0$

ie $\mathrm{Tor}^{\mathbb{Z}[b_1\dots b_{n-1}]}[\mathbb{Z},\mathbb{Z}]$, calculate via Koszul complex

More generally $\mathrm{Tor}^{\mathbb{Z}[b_1\dots b_{n-1}]=A}(\mathbb{Z}, M)$ where $b_i \to 0$ in $\mathbb{Z}$ factor:

need resolution $P_\bullet \to \mathbb{Z}$, take homology of $P_\bullet \otimes_A M$

Koszul $P_\bullet = A \otimes \Lambda^\bullet[\alpha_1,\dots \alpha_{n-1}]$ $\quad d\alpha_i = b_i \quad d(b_i) = 0$ – a DGA

$P_\bullet \otimes_A M = \Lambda^\bullet[\alpha_1,\dots\alpha_{n-1}] \otimes M \quad d(\alpha_i \otimes m) = b_i m$

In our case $P_\bullet \otimes_A \mathbb{Z} = \Lambda^\bullet[\alpha_1 \dots \alpha_{n-1}] \quad d\alpha_i = 0$

Seq Collapses, this $= K_* SU(n) = \Lambda^\bullet[\alpha_1 \dots \alpha_{n-1}] \quad \alpha_i \in K_1$

Twisted version: twist of $k \in H^3(SU(n);\mathbb{Z})$

$$\begin{tikzcd} \mathbb{C}P^{n-1} \subset \Omega SU(n) \arrow[r, hook] & PSU(n) \ni x \\ PSU(n) & \end{tikzcd}$$

$k \leftrightarrow k^{th}$ power of tautological line bundle of $\mathbb{C}P^{n-1}$ in "intersection". $\Omega SU(n)$.

K-cohomology

$$\begin{tikzcd} L^k & 1 \arrow[l, mapsto] \\ K^0 \mathbb{C}P^{n-1} & K^0(PSU(n)) = \mathbb{Z} \arrow[l] \\ & K^0(PSU(n)) = \mathbb{Z} \arrow[u] \arrow[ul] \end{tikzcd}$$

$1 \mapsto 1$

$$\begin{tikzcd} b_i \arrow[rr, mapsto] \arrow[d, mapsto] & & \binom{k}{i} \\ b_i & K_0(\mathbb{C}P^{n-1}) \arrow[r] \arrow[d] & \mathbb{Z} \\ 0 & \mathbb{Z} & \end{tikzcd}$$

$\Rightarrow K_*^\tau(SU(n))$ = homology of $\Lambda[\alpha_1, \dots \alpha_{n-1}]$, $d\alpha_i = \binom{k}{i}$

Let $d_n = GCD(\binom{k}{1}, \dots, \binom{k}{n-1})$

choose new basis $\alpha_1' \dots \alpha_n'$, $d\alpha_1' = d_n$, $d\alpha_i' = 0$

$\Rightarrow K_*^\tau SU(n) = \mathbb{Z}/d_n \otimes \Lambda[\alpha_2' \dots \alpha_{n-1}']$

C. Douglas applies this technique (using Bott-Samelson) to compute $K_*^\tau(G)$ G simply conn, simple

$$K_*^\tau(G) = \mathbb{Z}/d(G,k) \otimes \Lambda^\cdot[\alpha_1 \ldots \alpha_{r-1}] \qquad d \text{ integer}, \quad r = \text{rank}$$

$$\tau = k \in H^3(G;\mathbb{Z})$$

Künneth spectral sequence approach (V. Braun, needs hypothesis on Verlinde algebra)

<u>equivariant</u> $K^G_{\tau+*}(\underset{\text{left}}{G} \times \underset{\text{adjoint action}}{G}) = K_{\tau+*}(G)$

s.s. $\Uparrow$

$$\mathrm{Tor}^{K_G(pt)}(K_G(\overset{\text{left}}{G}), K_G(\overset{\text{adjoint}}{G})) = \mathrm{Tor}^{R(G)}(\mathbb{Z}, V) \quad \text{Verlinde algebra}$$

$\Rightarrow d(G,k) = GCD(\dim \rho_i)$ ρ_i = generators of Verlinde ideal

... need Verlinde ideal to be generated by regular sequence.

<u>Another technique</u>: equivariant version $\quad \mathrm{Tor}^{K_*^G(\Omega G)}(R[G], R[G]) \Rightarrow K^G_{\tau+*}(G)$

example $G = SU(2)$ $\quad K_*^G(\Omega G) = R[G][a,b]/a^2 - Vab - b^2 - a = A$

V = defining rep

$$A \xrightarrow{a,b \mapsto 0} R[G]$$

$$A \longrightarrow ?$$

$a \mapsto \rho_{k+1}$

$b \mapsto \rho_{k+1}\rho_k$

ρ_k irrep of high weight k.

Method of Tate for calculating Tor : Ill J. Math Vol 1, 1957

$A = \mathbb{Z}[a,b]/f(a,b)$, Calculate $\mathrm{Tor}^A(\mathbb{Z}, M)$ where $a,b \mapsto 0 \in \mathbb{Z}$

Could try $\tilde{P}_\cdot = A \otimes \Lambda[\alpha_1, \alpha_2]$ $\quad d\alpha_1 = a \quad d\alpha_2 = b$

– but this is not a resolution of $\mathbb{Z}$

Write $B = \mathbb{Z}[a,b]$! $\quad B \otimes \Lambda.[\alpha_1,\alpha_2] \xhookrightarrow{\cdot f} B \otimes \Lambda.[\alpha_1,\alpha_2] \twoheadrightarrow A \otimes \Lambda.[\alpha_1,\alpha_2]$

long exact seq in homology: ~~map~~ get $H_0\tilde{P} = H_1\tilde{P} = \mathbb{Z}$

ex: $f(a,b) = a^2 - ab \Rightarrow A$, gen. by $a\alpha_1 - a\alpha_2 = x$

Tate's resolution: $A \otimes \Lambda^\cdot[\alpha_1, \alpha_2] \otimes \Gamma^\cdot[\beta]$

$\Gamma[\beta]$ = divided power algebra, basis $\beta^{(n)}$ $\left[\text{" } \frac{\beta^n}{n!} \text{ "}\right]$

$\beta^{(n)}\beta^{(m)} = \binom{n+m}{n}\beta^{(n+m)}$

$d\beta = x$, $d\beta^{(n)} = \beta^{(n-1)}x$.. get effective resolution..

$|\beta| = 2$ ..

.... $\Rightarrow$ in our case all higher $\mathrm{Tor}^i = 0$, $K^G_{E=0}(G) = \frac{R[G]}{(\rho_{k-1}^2, \rho_k\rho_{k-2})}$

- not regular sequence...

Clebsch-Gordan $\rho_k = V\rho_{k-1} - \rho_{k-2}$ ... "Euclid's algorithm" for division $\Rightarrow \rho_k, \rho_{k-1}$ relatively prime, only need ρ_{k-1} to generate ideal.