M. Hopkins - Topological Modular Forms & the Theorem of the Cube 5/1/95

(w. Ando, Strickland) <u>Elliptic Cohomology</u>: main input is the (complex) Witten genus - associated to the characteristic series $\frac{z}{1-e^{-z}} \frac{\prod(1-q^n)^2}{\prod(1-q^n e^z)(1-q^n e^{-z})}$

$\nearrow$ operator $\longrightarrow$ q-powers of tangent bundle

M^{2n} stably almost complex $\longmapsto f_M(q)$ analytic function. $q = e^{2\pi i \tau}$

Witten: $c_1 = c_2 = 0$, then f_M is modular $f_M(\frac{-1}{\tau}) = (\tau)^n f_M(\tau)$ (full modular group.) Witten genus is a cobordism invariant

$MU\langle 6\rangle_* \longrightarrow$ Modular forms ($c_1=c_2=0$)

Would like to think of this as transformation of cohomology theory $\Rightarrow$ need to have notion <u>for families</u>.

want to write $MU\langle 6\rangle \longrightarrow E$, some cohomology theory. (manifolds moving in families/cobordisms, get cohomology with these coefficients.)

<u>Problem</u> Describe the set of multiplicative transformations $MU\langle 6\rangle \to E$

Def/~~Abuse~~ of terminology - E is complex ~~[illegible]~~ orientable (spectrum) if

- E is multiplicative, $E^*(pt)$ is commutative & concentrated in even degrees
- $E^{-2}(pt)$ contains a unit
- $\exists$ a class $x \in \tilde{E}^0 \mathbb{C}P^\infty \to \tilde{E}^0 S^2 = E^{-2}(pt)$ mapping to a unit. ($\tilde{E}$ - reduced)

i.e. $I = \tilde{E}^0(\mathbb{C}P^\infty)$, then I/I^2 (in unreduced) is iso to $\tilde{E}^0(S^2)$.

If instead of $MU\langle 6\rangle$ we take $MU \xrightarrow{\phi} E$, $\Longleftrightarrow$ genera: power series in 1 variable, $M^{2n} \mapsto \phi(M^{2n})$: ϕ is gotten by a stable exponential char. class on tangent bundle $\Rightarrow$ (splitting principle) element of BS^1 - canonical line bundle (determined by value on line bundles.)

$MU\langle 6\rangle$ - can't test on canonical line bundle (invariant def only for $c_1=c_2=0$).

- test on bundle on $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty$, $(L_1-1)(L_2-1)(L_3-1) = -\Lambda_{-1}(L_1+L_2+L_3)$ - get cohomology of Thom complex over product of three $\mathbb{C}P$'s - get power series in <u>three</u> variables.

<u>Cubical Structures</u> (Breen) G abelian group, $\begin{matrix} L \\ \downarrow \\ G \end{matrix}$ line bundle

$\Rightarrow$ new line bundle $\Theta(L)$ over G^3, $\Theta(L)_{x,y,z} = \dfrac{L_{x+y+z} \otimes L_x \otimes L_y \otimes L_z}{L_{x+y} \otimes L_{x+z} \otimes L_{y+z} \otimes L_e}$

- the second difference operator, testing for quadratic relations - kills any quadratic functor on line bundles. $H^2(\mathbb{Z})$ is a quadratic functor - for any bundle over a <u>topological space</u> $\Theta(L)$ will be canonically topologically trivial

<u>Def</u> A cubical structure on L is a section s of $\Theta(L)$ s.t.

1. $s(e,e,e) = 1$ (canonically a trivial line) (rigid)
2. $s(\sigma(x), \sigma(y), \sigma(z)) = s(x,y,z)$ $\sigma \in \Sigma_{\{x,y,z\}}$
3. $s(w+x,y,z)\, s(w,x,z) = s(w,x+y,z)\, s(x,y,z)$

Interpretation: first define $\Lambda(L)_{x,y} = \frac{L_{x+y} \otimes L_e}{L_x \otimes L_y}$ over $G \times G$.
(fixing a point in one G, get extension of G by $\mathbb{G}_m$, S is a cocycle giving the extension.) G an elliptic curve: $G \times G$ has canonical (Poincaré) line bundle (selfdual).

E complex orientable, G = the formal group of E. $\mathcal{O}_G = E^0(\mathbb{C}P^\infty)$
– $\mathbb{C}P^\infty$ is a group since it classifies lines: $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying tensor product: get $\mathcal{O}_G \to \mathcal{O}_G \hat{\otimes} \mathcal{O}_G$.
$L = \mathcal{O}_G(-e)$ ring of functions vanishing at the identity
$\Gamma(L) = \tilde{E}^0(\mathbb{C}P^\infty)$

<u>Theorem</u> (Via a map to be made specific) <u>the set $Mult(MU\langle 6\rangle, E)$ is isomorphic to the set of cubical structures on L.</u>
Theorem of the cube: G abelian variety, L alg. line bundle then $\Theta(L)$ is trivial.
(algebrogeometric version of topological classification of line bundles by $H^2(\mathbb{Z})$ – line bundles are quadratic). Thus L has a <u>unique</u> cubical structure – constant section with fixed value at identity.

<u>Def</u> An <u>elliptic spectrum</u> is a complex-orientable E with an elliptic curve $\mathbb{E}$ over $E^0(pt)$ and an isomorphism $t: \hat{\mathbb{E}} \xrightarrow{\sim} G$.
These form a category: maps of spectra + maps of elliptic curves.
<u>Def</u> for E elliptic, let $\sigma_E: MU\langle 6\rangle \to E$ correspond to the unique cubical structure on $\mathcal{O}_{\mathbb{E}}(-e)$
<u>Theorem</u> The (really neat) σ-orientation is <u>modular</u>, in the sense that given a map $E \to F$ of elliptic spectra the following diagram commutes

$$\begin{array}{ccc} MU\langle 6\rangle & \xrightarrow{\sigma_E} & E \\ & \searrow^{\sigma_F} & \downarrow \\ & & F \end{array}$$

EX Pick $\omega_1, \omega_2 \in \mathbb{C}$, $\mathbb{Z}$-linearly independent, $\mathbb{E} = \mathbb{C}/\langle\omega_1, \omega_2\rangle$
$E\langle\omega_1, \omega_2\rangle = H^*(-, \mathbb{C})[u, u^{-1}]$, $|u| = 2$.
The formal group is naturally $\mathbb{C} \to \mathbb{E} = \mathbb{C}/\langle\omega_1, \omega_2\rangle$. $E\langle\omega_1,\omega_2\rangle$ depends only on the lattice – $SL_2(\mathbb{Z})$-invariant
If $\lambda \in \mathbb{C}^*$ $\mathbb{C}/\langle\omega_1, \omega_2\rangle \xrightarrow{\lambda} \mathbb{C}/\langle\lambda\omega_1, \lambda\omega_2\rangle$
$\lambda: E\langle\omega_1,\omega_2\rangle \leftarrow E\langle\lambda\omega_1, \lambda\omega_2\rangle$, $\lambda u \leftarrow u$.

$$\begin{array}{ccc} [M^{2n}] \in MU\langle 6\rangle_{2n} & \longrightarrow & E\langle 1, \frac{a\tau+b}{c\tau+d}\rangle \\ & \searrow & \uparrow \cdot(c\tau+d) \\ \searrow & & E\langle 1,\tau\rangle = E\langle c\tau+d, a\tau+b\rangle \\ f_M(\tau)u^n \in & & \end{array} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

So $f_M\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-n} f_M(\tau)$.

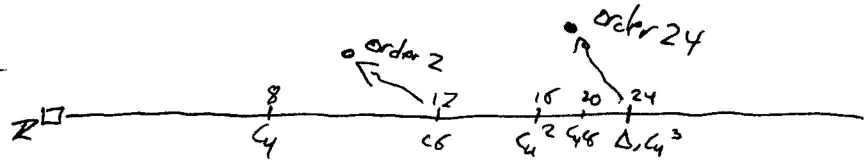
This map indeed corresponds to the Witten genus
~ Elliptic cohomology is really this whole family of elliptic cohomology theories

$MV\langle 6\rangle \xrightarrow{?} \underleftarrow{\text{homotopy limit}}_{E \text{ elliptic}} E = eo_2$ (a spectrum)

There's a spectral sequence from $\begin{pmatrix}\text{derived functor}\\ \text{of modular forms}\\ \text{over } \mathbb{Z}\end{pmatrix} \rightsquigarrow eo_2^*(pt)$

Ring of modular forms over $\mathbb{Z}$ - computed by Tate (0th derived functor) as
$\mathbb{Z}\,[c_4, c_6, \Delta]/c_4^3 - c_6^2 = 1728\,\Delta$.

Structure here accounts for all homotopy groups of spheres dimension up to 10



- building up a copy of orthogonal K-groups of spheres

- only get $24\Delta, 12\Delta^2, 8\Delta^3$ etc : $\frac{24}{(24,k)}\Delta^k$
Don't get pure Δ til dim 576 pick up Δ^{24} : periodic cohomology with period 576.

eo_2 is topological modular forms - do inverse limit as homotopy limit ~~as op~~, get wierd derived functors as opposed to usual

what's wrong with Δ? even unimodular lattices exist only in dim $8 \mid n$.

Get theta function $\Theta_K(q) = \sum_{0\neq \lambda\in K} q^{\frac{|\lambda|^2}{2}} = 1 + aq + \ldots$ from K - a modular form of weight half dim K. Lattice in dim 24 would be combo of Δ, c_4^3.

a - number of shortest vectors in lattice.

$c_4^3 = (1+240q)^3 \quad \Delta = q + \ldots$ so $\underline{a}$ determines the combination of c_4^3, Δ ;
but all a's for 24-dim unimodular lattices are divisible by 24 - Δ doesn't arise but $24\,\Delta$ does.

Do lattices give elements in elliptic cohomology? That would then imply which modular forms arise..

These derived functors correspond (over $\mathbb{Z}$) to higher cohomologies of $SL_2\mathbb{Z}$ acting on V - correspond to automorphism groups of supersingular curves. 24 - order of double cover of tetrahedron group (supersingular at prime 2.)

Borcherds proves a strong congruence for θ's of unimodular lattices - which allows for the possibility of ~~[illegible]~~ lattices giving elements.