M. Hopkins – Formal Groups arising from Plane Curves 11/4/96

Formal Group laws $/R$ $\longleftrightarrow$ cobordism theory invariants $MU_* \to R$

$\longleftrightarrow$ generalized homology, $MU \to E$, $E_*(pt) = R$

$X +_F Y$ commutative, identity 0, associative

If R is a $\mathbb{Q}$-algebra $\Rightarrow$ F has a log $\ell(x) = x + \sum a_n \frac{x^n}{n}$

Now $MU_* \supset \mathbb{Z}[[\mathbb{CP}^1], [\mathbb{CP}^2], \ldots]$

iso after $\otimes \mathbb{Q}$.

Suppose $MU \xrightarrow{\phi} R$, R torsion free: can try just to specify all $\underset{m_{n+1}}{\overset{\shortparallel}{\phi([\mathbb{CP}^n])}}$ .. subtle when this extends to MU_*.

Thm (Quillen) $\varphi: \mathbb{Z}[[\mathbb{CP}^n]] \to R$ extends to MU_* iff

$x + \sum \varphi([\mathbb{CP}]^n) \frac{x^{n+1}}{n+1}$ is the log of a formal group

Look for formal groups giving rise to geometric invariants of manifolds.

$\mathbb{G}_a$ gives invariant counting number of pts of 0-dim manifold, nothing else

$\mathbb{G}_m \rightsquigarrow$ Todd genus; in general $\ell'(x)\,dx = \frac{dx}{F_2(x,0)}$, $F_2 = \partial_y F$.

$\varphi(\mathbb{CP}^n) = 1$.

Elliptic curves $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$ $\Rightarrow A$

– plane curve meeting every line in 3 points, ∞ is a triple tangent.

A is a group: colinear points add to 0, ∞ = identity

Choose a parameter near ∞, expand group law $\Rightarrow$ FGL.

If $\frac{1}{2} \in R$ $\Rightarrow$ $y^2 = x^3 + b_2 x^2 + b_4 x + b_6$,

$t = \frac{x}{y}$ is local param at ∞, $\ell'(t)\,dt = -\frac{1}{2}\frac{dx}{y}$

Formal gp with symmetry $\rightsquigarrow$ refined invariants.

e.g. $\mathbb{G}_m$ $x \longrightarrow [-1](x) = 1 - (1-x)^{-1}$ automorphism.

$MU \longrightarrow K$ K-theory, $M^n \longmapsto K(S^n)$

$K^0(S^n) = \mathbb{Z} \quad 0 \quad \underset{-1}{\mathbb{Z}} \quad 0 \quad \underset{1}{\mathbb{Z}} \quad 0 \quad \underset{-1}{\mathbb{Z}} \quad 0 \quad \underset{1}{\mathbb{Z}}$

$\Rightarrow$ $\mathbb{Z}/2$ action on K-theory

from inversion on $\mathbb{G}_m$.

MSU = complex manifold bordism group w/ $c_1 = 0$.

$$\begin{tikzcd} MSU \arrow[r] \arrow[dr] & MU \arrow[d] \\ & \mathbb{Z}/2 \text{ invariant } K \end{tikzcd}$$

Naively $\Rightarrow c_1(M) = 0$ $n \not\equiv 0(4)$ $\Rightarrow \varphi(M^n) = 0$

More refined - rather than take $K^0(S^n)^{\mathbb{Z}/2}$, take
$K^{\mathbb{Z}/2}$ = KO theory $\quad \mathbb{Z} \;\; \mathbb{Z}/2 \;\; \mathbb{Z}/2 \;\; 0 \;\; \mathbb{Z} \;\; 0 \;\; 0 \;\; 0 \;\; \mathbb{Z}$
$\Rightarrow$ $MSU \to KO$, read off
that $\phi(M^{8k+4}) \equiv 0 \;(2)$ & pick up new
mod 2 invariants ... $\to$ index of Dirac.

Elliptic curve case: symmetries $x \mapsto \lambda^2(x+r)$, $y \mapsto \lambda^3(y+sx+t)$
- get isomorphic curve with different a's, $\Rightarrow$
big group action
$\Rightarrow$ $\mathbb{Z}[a_1, \ldots a_6]^{\text{symmetries}}$ = Ring of modular forms $/\mathbb{Z}$
Taking more refined, homotopy version of invariants
$\Rightarrow$ topological modular forms, which maps to above,
isomorphism if we invert 6. $\Rightarrow$ cohomology theory eo_2.
Seems to be a map $MU\langle 6\rangle \to eo_2$ $\quad (c_1, c_2 = 0)$.

<u>New examples</u> $\quad C: \; y^{p-1} = x^p + a_1 x^{p-1} + \ldots + a_p$
(Hopkins - Mahowald)
<u>Theorem</u> If $l \equiv 1 \; (p-1)$ (think $l = p$), and
we're over a $\mathbb{Z}_{(l)}$-algebra R, Then $-\frac{dx}{(p-1)y} = l'(t)$,
$t = \frac{x}{y}$ is dlog of a formal group over R.
[say p is prime].
Symmetry: $\quad y \mapsto \lambda^p y, \quad x \mapsto \lambda^{p-1}(x+r)$
preserves formal group.
<u>Geometric explanation</u>: Jacobian J_C = free abelian gp on C / ∞ = id, colinear points sum to 0.

Classical theorems: 1. J_C is a smooth proj variety
2. $\dim J_C$ = genus $\frac{(p-1)(p-2)}{2}$
3. $T_e^* J_C$ (cotangent space at id.) = $H^0(\Omega^1_C)$

Basis of $H^0(\Omega^1_C)$: $\quad \frac{dx}{y}, \; \frac{x\,dx}{y^2}, \; \frac{dx}{y^2} \; \ldots \; x^{p-3}\frac{dx}{y^{p-2}} \; \ldots \; \frac{dx}{y^{p-2}}$.

If ξ is $(p-1)$ root of unity, $y \mapsto \xi \cdot y$, $x \mapsto x$ is
automorphism, quotient is $\mathbb{P}^1$.

$\Rightarrow$ action of μ_{p-1}. $\frac{dx}{y}$ multiplied by ζ^{-1} … $\frac{dx}{y^{p-2}}$ by $\zeta^{-(p-2)}$ etc. Try to do eigenspace decomposition:

Over a $\mathbb{Z}_{(l)}$ algebra, the formal completion of J_c depends only on l-torsion in J_c, ~~End~~ $(\widehat{J_c})$ is a module $/\mathbb{Z}_l$.

$l \equiv 1\ (p-1) \Rightarrow \mathbb{Z}_l$ contains $\frac{1}{p-1}$ & $(p-1)^{st}$ roots of 1.

$\rightarrow$ use projection operator to ζ^{-1} eigenspace of $\widehat{J_c}$

$\Rightarrow$ Cohomology theory EO_{p-1} (using fgl + symmetries) (at least completed version)

Open questions: is there a natural genus attached to these?

Space of these includes full formal deformation space of fgl of height $p-1$.