Namboodiri Lectures

2/26/00 M. Hopkins - Modular Forms & Algebraic Topology

$H^*(\mathbb{CP}^\infty; R) = R[x]$

Cell structure of $\mathbb{CP}^\infty$ (one cell in every even dimension) $\longleftrightarrow$ $f(x) \in H^*(\mathbb{CP}^\infty; R)$ is a <u>function</u> (on affine line over R)

<u>Modular forms</u> e.g. Eisenstein series wt $2k$

$$E_{2k} = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^{2k}}$$

converges absolutely $k>1$, E_2 quasi-modular

q-expansion $E_2 = (\text{const})(1+24\sum_{n\geq 1}\sigma_1(n)q^n)$

$\sigma_k(n) = \sum_{d|n} d^k$

$E_4 = \text{const}\,(1+240\sum\sigma_3(n)q^n) \leftarrow c_4$

$E_6 = \text{const}\,(1-504\sum\sigma_5(n)q^n) \leftarrow c_6$

Notice $\pi_{n+3}(S^n) = \mathbb{Z}/24$ $\quad \pi_{n+7}(S^n)=\mathbb{Z}/240$ $\quad \pi_{n+11}(S^n)=\mathbb{Z}/504$

$n>>0$. ... given in terms of Bernoulli numbers, as is image of J homomorphism in homotopy of spheres.

- think of elements in cohomology as functions.

Modular forms over $\mathbb{Z}$: q-expansion coeffs $\in \mathbb{Z}$.

e.g. c_4, c_6 (Eisenstein/constant), $\Delta = q\prod(1-q^n)^{24}$ (wt 12)

<u>Theorem (Tate)</u> Ring of modular forms $/\mathbb{Z}$ is given by

$$\mathbb{Z}[c_4, c_6, \Delta]/\, c_4^3 - c_6^2 = 1728\Delta.$$

<u>From lattices</u> $L = \mathbb{Z}^D, \langle,\rangle$ positive, even, unimodular $\det\langle e_i, e_j\rangle = 1$.

($\Rightarrow D \equiv 0\ (8)$). e.g. $E_8 = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{pmatrix}$ first example

$$\Theta_L(q) = \sum_{\ell\in L} q^{\frac12\langle \ell,\ell\rangle} = \sum_{n\geq 0} \underbrace{\#\{\langle \ell,\ell\rangle = 2n\}}_{L_n} q^n$$

modular weight $\frac12 D$

Tate $\Rightarrow$ first few L_n determine the rest

e.g. $\Theta_{E_8}(q) = c_4$ (wt 4 of form $1+\dots$)

- $L_n \sim n^{D/2-1}$ $\quad$ • $L_n \equiv 0\ (2)$

<u>Borcherds' congruence</u> Suppose $\dim L = 24k$,

$\Theta_L = x_0\Delta^k + x_1\Delta^{k-1}c_4^3 + \dots + x_k c_4^{3k}$

Then $x_0 \equiv 0\ (24)$.

$\Leftrightarrow$ $\frac{\Theta_L}{\Delta^k} = q^{-k} + \dots + a_0 + a_1 q + \dots$ & $a_0 \equiv 0\ (24)$

- main obstruction to bringing Θ functions into algebraic topology..
comes from forms on $SO(n+2,2)$

Today: 3 simpler proofs : think of fns as elements in some cohomology & of attaching maps. $\rightsquigarrow$ only prove mod 12, not mod 2^k...

(1) Weierstrass $\wp$-function $\wp(z,\tau) = \frac{1}{z^2} + \sum_{\substack{m,n \neq 0,0 \\ m,n \in \mathbb{Z}}} \frac{1}{(z-m\tau-n)^2} - \frac{1}{(m\tau+n)^2}$

- gives E_k's as z-coefficients.

characterize $\wp$:
- $\wp(z,\tau)\,dz^2$ invariant under $z \mapsto z+m\tau+n$ & $(z,\tau) \to \left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$ $(SL_2\mathbb{Z})$
- $\wp(z,\tau)(dz)^2 = \frac{1}{z^2}dz^2 + \cdots\cdots$ square residue at zero

$$\wp(z,\tau)dz^2 = \left(\sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} + \frac{1}{12} - 2\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}\right)\left(\frac{du}{u}\right)^2 \quad u = e^{2\pi i z}$$

What's special about Θ_L^2?

Ex $(c_4\,\wp(z,\tau) - \frac{1}{12}c_6)dz^6$ is integral. (240, 504 div. by 12)

But can't show $(\Delta^k \wp(z,\tau) - g)(dz)^{12k+2} \equiv 0\ (\mathbb{Z})$ for some modular g wt $12k+2$.

Borcherds mod 12 $\Longleftarrow$ $\boxed{\text{Proposition}}$ L even unimodular dim D - then $\exists$ modular g_L wt $D/2+2$ for which

$$(\Theta_L \cdot \wp(z,\tau) - g_L)(dz)^{D/2+2} \equiv 0 \ (\mathbb{Z}).$$

Proof In case $\exists\, \mu \in L$, $\langle\mu,\mu\rangle = 2$

Let $\Theta_L(z,\tau) = \sum_{l\in L} e^{(\pi i\langle l,l\rangle\tau + 2\pi i\langle l,z\rangle)}$ $z \in \mathbb{C}\otimes L$

(previous was "theta null" $\Theta_L(0,\tau)$)

Weierstrass σ: $\frac{\sigma(z)}{dz} = u^{\frac{1}{2}}(1-u^{-1})\frac{\prod_{n\geq 1}(1-q^n u)(1-q^n u^{-1})}{\prod_{n\geq 1}(1-q^n)^2}\frac{u}{du}$

Closest to periodic fn with only simple zero at lattice point : analog of $z-a$.

Θ-identities $\Rightarrow$ $(z \in \mathbb{C}$

$$\varphi(z,\tau)(dz)^{D/2+2} := \frac{\Theta_L(\mu z,\tau)}{\sigma(z,\tau)^2}(dz)^{D/2+2}$$

is invariant under $z \mapsto -z$, $z \mapsto z+m\tau+n$,

$z, \tau \mapsto \left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$ $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2\mathbb{Z}$

Furthermore $\varphi(z,\tau)(dz)^{\frac{\ell}{2}+2} = \left(\frac{\theta_L}{z^2} + \dots\right)(dz)^{\ell/2+2}$

$[\ \wp(z,\tau)\,\theta_L = \frac{\theta_L}{z^2} + \dots\ ]$

$[\varphi(z,\tau) - \theta_L \cdot \wp(z,\tau)]\,(dz)^{\ell/2+2}$ has no poles,
& lattice invariant $\Rightarrow$ constant in z,
call it $g_L(\tau)$.

$\theta_L \wp - g_L =$ our φ, which is integral, as needed $\blacksquare$

[need assumption on μ for transformation properties].

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(2) Algebraic POV $\quad \mathcal{M}$ = moduli stack of elliptic curves

identity $e \curvearrowright \mathcal{E} \downarrow p \ \mathcal{M}$ universal curve. ω = sheaf of invariant 1-forms on $\mathcal{E}$ (= cotangent along e), line bundle on $\mathcal{M}$.
— generated by dz

Note $H^0_{\mathcal{M}}(\omega^k) = \mathrm{Hom}_{\mathcal{M}}(1, \omega^k)$ sections of ω = modular forms of wt $k/\mathbb{Z}$

$\wp$-function: Short exact seq $[0 \to \omega^2 \to p_* \mathcal{O}(2e) \otimes \omega^2 \to 1 \to 0]$

Over $\mathbb{C}$, $\wp(z)dz^2$ gives a splitting
($p_*\mathcal{O}(2e)$: fns with at worst double pole, 2-dimensional)
$\mathrm{Hom}_{\mathcal{M}}(1, \omega^2)$ = mod form of wt 2 $= 0$ $\Rightarrow$ uniqueness of splitting.

Consider short exact above as $\nu \in \mathrm{Ext}^1_{\mathcal{M}}(1, \omega^2)$

Facts: • $\mathrm{Ext}^1_{\mathcal{M}}(1, \omega^2) = \mathbb{Z}/12 \cdot \nu$.
• $\Delta^k \nu \in \mathrm{Ext}^1_{\mathcal{M}}(1, \omega^{12k+2})$ also exact order 12.

Proposition $\quad L$ even unimodular $\Rightarrow \theta_L \cdot \nu = 0 \in \mathrm{Ext}^1_{\mathcal{M}}(1, \omega^{\ell/2+2})$

12 comes from order of Ext group (thinking of $\mathcal{M}$ as stack over $\mathbb{Z}$)
Proof of Ext : comes from studying suitable supersingular curves.
Ext splits away from supersingular locus (in char 2, 3)
eg in domain of q-expansion.

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(3) [Note only need to assume vector μ of $\langle\mu,\mu\rangle = 2$ mod 12..]

Topology POV $\quad$ Try to find a cohomology theory E with $E^0(S^{2n})$ = modular forms of wt n over $\mathbb{Z}$.

... best you can do is a spectral sequence
$\mathrm{Ext}^s(1, \omega^t) \Rightarrow E^0(S^{2t-s})$

(Hard to write short exact sequence in topology)

$\Rightarrow$ (maps between spheres) $\longleftrightarrow$ ($\mathrm{Ext}^s(1,\omega^t)$)

Hopf map $\nu: S^7 \to S^4 \longleftrightarrow \nu \in \mathrm{Ext}^1(1,\omega^2)$

L = lattice, V_L = any rep of $S^1 \otimes L$ torus • of virtual dim = dim L

• $c_1 \equiv 0\ (2)$ • $\frac{c_1^2 - 2c_2}{2} = \langle , \rangle \in H^4(BS^1 \otimes L) = \mathrm{Sym}^2(L^*)$

[V_L complex rep]

Θ_L in topology: Thom complex $(BS^1 \otimes L)^{V_L} \xrightarrow{\Theta_L(z,\tau)} E$

$$\begin{tikzcd} (BS^1\otimes L)^{V_L} \arrow[r, "\Theta_L(z,\tau)"] & E \\ S^{|V_L|} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Θ_L: element in E-cohomology of sphere is modular form.

We used z variable & order two point essentially...

Θ extends to elt in cohomology of Thom complex, new variable z gives new cells in Thom space

Borcherds follows from $S^{3+|V_L|} \xrightarrow{\nu} S^{|V_L|} \xrightarrow{\text{inclusion}} (BL \otimes S^1)^{V_L}$

$$\begin{tikzcd} S^{3+|V_L|} \arrow[r, "\nu"] \arrow[rr, bend right, "\text{trivial map } *"'] & S^{|V_L|} \arrow[r, "\text{inclusion}"] & (BL\otimes S^1)^{V_L} \end{tikzcd}$$

Easiest way to check is to pick $\mu \in L$ and map $S^1 \xrightarrow{1\otimes\mu} S^1 \otimes L$

Take $(1\otimes\mu)^* V_L = \langle\mu,\mu\rangle(\underset{\text{trivial}}{1} - \underset{\text{canonical line bundle}}{\mathbb{L}}) + \underset{\text{trivial}}{\dim L}$ as S^1-rep

So as long as $\langle\mu,\mu\rangle \equiv 2\ (2)$,

$\nu \cdot$ Thom class $= n \cdot$ Atiyah

$\mathbb{L}$ = canonical line bundle

$$\begin{tikzcd} & S^{-\langle\mu,\mu\rangle + \dim L} \arrow[d] & (BS^1)^{-\langle\mu,\mu\rangle \mathbb{L}} \arrow[r] & (BS^1 \otimes L)^{V_L} \arrow[d, "\Theta_L(z,\tau)"] \\ S^{3+|V_L|} \arrow[r, "\nu"] & S^{|V_L|} & & E \end{tikzcd}$$

show this is null

The 12 here is the J order of $\mathbb{L}$ on $\mathbb{CP}^2$

Hopkins–Mahowald

Hopkins–Greenlees: analog for higher dim abelian varieties.

Relation to Moonshine manifold: Spin manifold with Witten genus the theta function of Leech lattice.. hope monster acts on it.

M. Hopkins Namboodiri II - Alg. Topology & Modular Forms (Jt w/ Singer)

1930s Pontryagin - compute homotopy grps of spheres via framed cobordism

- $\pi_{n+1} S^n = \mathbb{Z}/2$ $n>>0$
- $\pi_{n+2} S^n = 0$ $n>>0$ — false... (actually $\mathbb{Z}/2$)

His argument: start with map

S^{n+2} Σ Riemann surface   . get framing of normal bundle from inverse image of disc

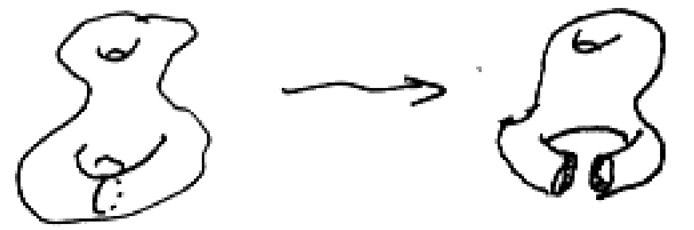
$\downarrow$

$S^n \supset D \ni$ pt regular value

Can collapse exterior of disc to point! map determined by geometry of framed normal bundle.

genus $\Sigma = 0 \Rightarrow$ map is null ($\pi_2 SO(N) = 0$!)

genus $= g$: try to lower genus



framed surgery

$\mathfrak{S}^?$ are framed, as is RS : need framings good to extend over boundary. Get obstruction $\varphi : H_1(\Sigma) \to \mathbb{Z}/2$ ($\mathbb{Z}/2$ = gp of framed 1-mflds)

Mistake: thought φ is linear. In fact quadratic,
$\varphi(x+y) - \varphi(x) - \varphi(y) = x \cap y$.

Arf φ detects nontrivial element in $\pi_{n+2} S^n$.

$\to$ look for quadratic function on middle cohomology of $4k+2$-manifolds (twice odd number) — Kervaire, Browder, E. Brown ($\leadsto$ 10-manifold without smooth structure!)

1860s Riemann: Σ RS of genus g. Theta characteristics:
$\mathcal{L}$ s.t. $\mathcal{L}^2 \cong \omega$ hol 1-forms

Parity of $\mathcal{L}$: $q(\mathcal{L}) = \dim H^0(\mathcal{L})$ mod 2

If $\alpha^2 = \mathcal{O}$, $\mathcal{L} \otimes \alpha$ another θ-characteristic.

- $q(\mathcal{L})$ constant under deformations.
- $q(\mathcal{L} \otimes \alpha) - q(\mathcal{L})$ quadratic function of $\alpha \in H^1(\Sigma, \mathbb{Z}/2)$ with bilinear form the cup product.

($\dim H^0(\mathcal{L})$ = order of vanishing of Θ $\Rightarrow$ sign = sign in functional eqn for Θ ......)

~1970: Atiyah, Mumford : relate these quadratic functions
Θ-char $\leftrightarrow$ spin structure, get q = index of $\not{D}_L$.
$\to$ spin cobordism invariant $\longleftrightarrow$ framed cobordism invariant.

	Topology	Algebra
dim 2 :	Pontryagin	Riemann's parities
$4k+2$:	Kervaire	$\boxed{?}$

Singular curves : Harris extends Riemann .. works for families, & with boundary, etc.
Kervaire invt in dim 6 appears in EO theory -- relation to elliptic cohomology... Witten '97-8 6 dim version of theta function: 5-brane partition function in M-theory.

Sketch of proof of Riemann Given $\Sigma, \mathcal{L}$ $\Rightarrow$ $\det \mathcal{L} = \det H^0(\mathcal{L}) \otimes \det H^1(\mathcal{L})^*$
Serre duality $\Rightarrow$ $\det(\omega \otimes \mathcal{L}^{-1}) \simeq \det(\mathcal{L})$

$\mathcal{L}^2 \simeq \omega \Rightarrow$ $\omega \otimes \mathcal{L}^{-1} \simeq \mathcal{L}$, get another isomorphism

$\det \mathcal{L} \xrightarrow{\sim} \det \omega \otimes \mathcal{L}^{-1}$ $\quad \Lambda(H^0(\mathcal{L}) \oplus H^1(\mathcal{L})^*)$

$\downarrow$ Θ-char $\quad$ is

$(-1)^{\dim H^0} \to \det \mathcal{L}$ $\quad \Lambda(H^0(\mathcal{L}) \oplus H^0(\mathcal{L}))$ switch factors

- so Riemann's parity is isom of $\det \mathcal{L}$, parity must be constant.
Must check this is independent of choice of $\mathcal{L}^2 \simeq \omega$
- the number $g-1 = \frac{1}{2} c_1$ plays a role.

More generally have family of R.S. $\Sigma \to E \leftarrow \mathcal{L}$, $E \downarrow S$, $\det_{E/S} \mathcal{L}$ line bundle

$\mathcal{L}^2 = 1$ classified by $H^1(E, \mathbb{Z}/2)$.

Now generalize to middle cohomology $2n = 4k+2$, $\varphi: H^n(M, \mathbb{Z}/2) \to \mathbb{Z}/2$
Replace $H^1(E; \mathbb{Z}/2) \dashrightarrow H^n(E; \mathbb{Z}/2)$
$\mathcal{L} \in H^2(E; \mathbb{Z}/2) \rightsquigarrow H^{n+1}(E, \mathbb{Z})$

Need a kind of $\det : H^{n+1}(E) \to H^2(S)$
& a category with objects classified by $H^{n+1}(E)$

$\dim E/S = 2$: $c_1(\det \mathcal{L}) = \int_{E/S} \frac{x^2 - xc_1}{2}$
$x = c_1(\mathcal{L})$ $c_1 = c_1(E/S)$

$\dim E/S = 2n$: want $\int_{E/S} \frac{x^2 - x\lambda}{2}$

λ some char. class: should be in $H^{n+1}(E;\mathbb{Z})$ mapping down to Wu class $\nu_{n+1} \in H^{n+1}(E;\mathbb{Z}/2)$

- λ won't exist for any family ...
RS case: need Spin^c structure on relative tan bundle, comes from orientation.

Two steps: 1. Produce the topological "class"
2. Refine to geometric setting.

For step 2: work with Deligne-Beilinson cohomology: add connection to line bundle ... at least need a form with appropriate de Rham class

M = smooth manifold. On cochains:

$$\begin{array}{ccc} C(n)^*(M) & \longrightarrow & \Omega^*(M)_{\geq n} \\ \downarrow & & \downarrow \\ C^*(M;\mathbb{Z}) & \longrightarrow & C^*(M;\mathbb{R}) \end{array}$$

homotopy pullback diagram.
— lift integer cohomology to n-forms $\iff$ quantize n-forms (integral periods).

$\Rightarrow$ integral cochain + form + real cochain whose coboundary is the difference.

$$H^k C(n)^*(M) =: H(n)^k(M)$$

$$H^{n-1}(M)\otimes \mathbb{R}/\mathbb{Z} \hookrightarrow H(n)^n(M) \twoheadrightarrow A^n(M)$$

$\{(x,\omega)\ x \in H^n(M;\mathbb{Z}),\ \omega \in \Omega^n_{cl},\ [\omega] = x \in H^n(X;\mathbb{R})\}$

- Everything in sight is a ring $\Rightarrow$ $C(n)^*(-)$ multiplicative.
- Thom isomorphism, integration etc work as in ordinary cohomology.
- Construction makes sense for generalized cohomology theories E: replace cochains with model for maps into E-M space $\Longrightarrow$

$$\begin{array}{ccc} E(n)^*(M) & \longrightarrow & (\Omega^*(M)\otimes E_*)_{\geq n} \\ \downarrow & & \downarrow \\ E^M & \longrightarrow & (E\otimes\mathbb{R})^M \end{array}$$

$E_* = E(pt)$

real localization of E

think of this as finding forms to represent E, or take forms & quantize to get periods in E cohomology of M.

$\Rightarrow$ Differential E-cohomology : useful for Ramond-Ramond fields (Freed-Hopkins: use differential K-theory).

$M^{2n} - E$ n odd.
$\quad |$
$\quad S$

Let $Z^{n+1}(E)$ = category with objects: closed elements in $C(n+1)^{n+1}(E)$, morphisms: elements in $C(n+1)^n / \delta C(n+1)^{n-1}$.

Given cochain complex can always form category, with objects classified by H^{n+1} : objects = $(n+1)$ cocycles, map $a \to b$ is $c \in C^n$, $\delta c = b - a$, can mod out by δC^{n-1}.
— fundamental groupoid of space of maps into Eilenberg-MacLane space of C^n were cochain complex...
Aut (objects) $= H^n$.

$n=1$: $Z^2(E)$ equivalent to category of $U(1)$ bundles with connection.

Proposition Associated to each integral Wu structure (integral lift λ of $\nu_{n+1}(E/S)$ -- get e.g. from Spin structure) is a "det" functor $g: Z^{n+1}(E) \to Z^2(S)$ with properties analogous to det. ... enough to prove parity statement.

Arf invariant lives on set of choices of $\frac{1}{2}\lambda$.
- λ analogous to ω

Ex M^{2n} manifold, n odd & no torsion in $H^{n+1}(M;\mathbb{Z})$

$H^n(M)\otimes\mathbb{R}/\mathbb{Z} \to H(n+1)^{n+1}(M) \twoheadrightarrow \Omega^n_{cl}$, $\mathbb{Z}$ periods
$\quad \wr$
$H^n(M, \mathbb{R}/\mathbb{Z})$

Choose a metric on $M \Rightarrow *: \Omega^n(M) \circlearrowleft$, $*^2 = -1$
$\Rightarrow$ complex structure on $\Omega^n(M)$
$\Rightarrow$ complex structure to harmonic n-forms $\simeq H^n(M;\mathbb{R})$
$\Rightarrow$ complex structure on $H^n(M, \mathbb{R}/\mathbb{Z}) = J$

This satisfies Riemann relations $\Rightarrow H^n(M;\mathbb{R}/\mathbb{Z})$ is abelian variety, $\cup$ gives principal polarization. This theory gives symmetric divisor giving polarization $\Rightarrow$ theta function.

The proposition gives, for each choice of $\frac{1}{2}\lambda$ ("Θ-char") a holomorphic line bundle (symmetric divisor) giving rise to the polarization.
$\rightarrow$ theta function, giving the Kervaire invariant.

Witten: M 6-mfld, partition fn. for 5-brane effective action. Must make topological choice — Wu structure $\leftrightarrow \omega^{\frac{1}{2}}$.
- appears as anomaly, which need to cancel to get path integral.
Differential K-theory gives coupling of RR field to brane.

T.S. Eliot: Poetry is the field created by the proximity of elements.

In families get $Z^3(S)$ integrality: gerbes on S.. but really measured not by cohomology (D-B) but by "Anderson dual of sphere" _ new torsion phenomena....

We get Θ divisor in J, geometry must be sensitive measure of metric on M.. are these Jacobians? what do they look like?

$q(\lambda - x) \simeq q(x)$ canonical ("Serre duality")
Arf invariant comes up as sign of automorphism like parity
Kervaire $\longrightarrow$ symmetric line bundle over parameter space.

M. Hopkins

Namboodiri III

Moonshine

2/29/00

Irreps of monster group have dimensions
1 196883 21296876

Modular function $j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots$

- Conway-Norton: $\exists$ reps of monster corresponding to each of these coefficients ... (FLM, Borcherds) ..
- Hirzebruch's book "Manifolds & modular forms"

Witten genus $\Phi_W(M) = \hat{A}(M; \bigotimes_{n\geq 1} \mathrm{Sym}_{q^n} \overline{T_{\mathbb{C}}})$

$T_{\mathbb{C}}$: $\mathbb{C}$-tangent bundle of M, $\overline{T_{\mathbb{C}}} = T_{\mathbb{C}} - \mathbb{C}^{\dim M}$ ... stable bundle.
(M spin manifold).

If char class $\frac{1}{2}p_1(M) = 0 \Rightarrow \Phi_W(M)$ is modular form of wt $\frac{1}{2}\dim M$.

Q: Is there a spin manifold M^{24}, $\frac{p_1}{2}(M) = 0$ with
$\Phi_W(M) = \Delta \cdot (j - 744)$ Δ discriminant
$\Rightarrow j - 744 = \hat{A}(M, \otimes \mathrm{Sym}_{q^n} T_{\mathbb{C}})$

[$\mathrm{Sym}_q V = \sum \mathrm{Sym}^n V \cdot t^n$]

Hope: Monster acts on M, & $\otimes \mathrm{Sym}_{q^n} T_{\mathbb{C}}$ realizes the moonshine module.

- Mahowald-Hopkins construct M (without monster action) out of Poincaré complex of cells...
- Part of bigger story in elliptic cohomology.
- Borcherds congruence one of main obstructions to good construction

Elliptic Cohomology (Topological Modular Forms "TMF")

Main property: spectral sequence $\mathrm{Ext}^s_{\mathcal{M}}(1, \omega^n) \Longrightarrow \pi_{2n-s}(TMF)$
$= TMF^0(S^{2n-s})$
$\mathcal{M}$ = moduli stack of elliptic curves

[$\mathrm{Ext}^0_{\mathcal{M}}(1, \omega^n)$ = modular forms of wt. n]

Edge homomorphism: $\pi_* TMF \longrightarrow \mathbb{Z}[c_4, c_6, \Delta]/c_4^3 - c_6^2 = 1728\Delta$
$\mathbb{Z}$-modular forms.

.. S.S doesn't degenerate, but well understood... relates much of homotopy of spheres to geometry of modular forms.

This edge map is not onto, image has basis

$2c_4^a c_6 \Delta^b$, $c_4^a \Delta^b$ $a > 0$, $\frac{24}{(24, b)} \Delta^b$

- there's a differential $d_4 \Delta = \bar{\kappa} \in Ext^4(1, \omega^{?})$ of order 24, which is a derivation.

Where do these arise in nature? e.g. Θ-fns of an unimodular lattice # of vectors of length 2 divisible by 24...
Borcherds congruence $\Rightarrow$ $\Theta_L \in im(\pi_* TMF)$ for any L even unimodular. ... more refined than just any modular form.
Would like to go from lattice to element in elliptic cohomology of a sphere.

<u>Witten genus</u>: map $MO\langle 8\rangle \longrightarrow TMF$
(bordism of spin manifolds with $\frac{p_1}{2} = 0$)... map not quite constructed, but this is what it should do.
<u>Thm</u> (Mahowald-(+)) $\pi_* MO\langle 8\rangle \twoheadrightarrow \pi_* TMF$

- immediately implies existence of moonshine manifold:
$j = c_4^3/\Delta$ so $\Delta(j-744) = c_4^3 - 744\Delta$, $744 \equiv 0$ (24).
(in dim 24 $\pi_* MO\langle 8\rangle$ has rank 4, $\pi_* TMF$ rank 2).

(Corollary) <u>Proposition</u> M 24-dim spin manifold, $\frac{1}{2}p_1 = 0 \Rightarrow$
Morita-Schwinger operator $\hat{A}(M, T_{\mathbb{C}}) \equiv 0$ (24) Dirac index
[elliptic cohomology version of Rokhlin's theorem]

<u>Cor.</u> • <u>Thus</u> for every even unimodular lattice L there is an $MO\langle 8\rangle$-manifold M_L with $\Phi_w(M_L) = \Theta_L$.

$\rightsquigarrow$ look for topological theory of Θ_L, realizing M_L.
imagine it somehow functorial in lattice, or for some extra data as well.
Monster built from Aut(Leech lattice)
$\rightsquigarrow$ take L = Leech lattice, get monster action out of naturality of $L \mapsto M_L$ .... at least maximal proper subgrp.
$\Phi_w(M_L) = \Theta_L = 1 + O(q^2)$
L only even unimod with no vectors $\langle v,v\rangle = 2$
$\Rightarrow \Theta_L = c_4^3 - 720\Delta$

$\frac{\Phi_w(M)}{\Delta} = j - 720$ (not 744: 24 instead of 0...)
$\Delta(j-744)$ is not Θ-fn of lattice.

Algebraic theory of theta functions

A = abelian variety of rank g, $\mathcal{L}$ = line bundle$/A$, symmetric giving rise to polarization

Symmetric: $A \xrightarrow{(-1)} A$, $(-1)^*\mathcal{L} \simeq \mathcal{L}$.

fix triv of fiber $\mathcal{L}_e = \mathbb{1}$.

- $A \to \mathrm{Pic}\, A$ $\quad a \mapsto t_a^*\mathcal{L} \otimes \mathcal{L}^{-1}$ is isomorphism.

Ex A = elliptic curve, $\mathcal{L} = \mathcal{O}(e)$ simple pole at identity

$\Rightarrow A \simeq A^\vee$, symmetric but need $\mathcal{L} = \mathcal{O}(e) \otimes \omega^{-1}$

to set $\mathcal{L}_e = \mathbb{1}$ (inverse of residue)

- $\dim H^0(\mathcal{L}) = 1$, $\quad H^i(\mathcal{L}) = 0$ $i > 0$.
- Formule Clé (Moret-Bailly, Faltings-Chai):

$$H^0(\mathcal{L})^8 \simeq \omega_g^{-4}, \quad \omega_g = \text{invariant } n\text{-forms on } \mathcal{A}_g = \Lambda^g(\text{invt 1-forms})$$

- identification of line bundles over moduli of abelian varieties.

Isom is unique up to sign! but not constructive

Ex $g=1$ $\mathcal{L} = \mathcal{O}(e) \otimes \omega^{-1}$. $H^0(\mathcal{L}) = \omega^{-1}$.

Formule Clé $\Rightarrow$ $\omega^{-8} = \omega^4 \Rightarrow 1 \simeq \omega^{12}$; multiplication by Δ ..

But Δ not topological modular form ... will be problem.

Borcherds congruence supposed to get us around this...

L even unimodular lattice $\langle\,,\rangle$

$\Rightarrow (\mathcal{A}_g, \mathcal{L}) \xrightarrow{\otimes L} (\mathcal{A}_g \otimes L, \mathcal{L} \otimes L)$

$f: L \to \mathbb{Z} \Rightarrow A \otimes L \xrightarrow{f} A$, $\Rightarrow f^*\mathcal{L}$ pull back

Pullback line bundles are symmetric: $\sigma := (-1) : L \to L$

$$\begin{tikzcd}
\mathbb{Z} & & & & \\
\mathbb{Z}[L^*] \arrow[u] \arrow[rrr] & & & \mathrm{Pic}\, A) \ni \mathcal{L} \otimes L & \\
I \arrow[u, phantom, "\cup"] \arrow[r] & I/I^3 \arrow[urr, "\text{Thm of the cube}"] \arrow[rr] & & I/(I^3, (1-\sigma)) \arrow[u] \simeq (\mathrm{Sym}^2 L)^\vee \ni \langle\,,\rangle &
\end{tikzcd}$$

augment ideal

$(\mathrm{Sym}^2 L)^\vee \cong \Gamma_2(L^*)$ second divided power

$\langle\,,\rangle$ even unimod

E = elliptic curve $\rightsquigarrow$ element of $\omega^{D/2}$:

$\mathcal{L} = \mathcal{O}(e) \otimes \omega^{-1} \Rightarrow$ line $H^0(L \otimes E, L \otimes \mathcal{L})$

$\mathcal{L}$, $\mathcal{L}\otimes L$ have canonical triv at identity $\Rightarrow$

$H^0(L\otimes E; L\otimes \mathcal{L}) \xrightarrow{\text{eval at } 1} 1$

$\Leftrightarrow$ $1 \longrightarrow H^0(L\otimes E, L\otimes\mathcal{L})^{-1}$ dual line

Modly $H^0(\mathcal{L})^{-1} \simeq \omega_g^{\frac{1}{2}}$ formule clé

So $1 \longrightarrow H^0(L\otimes E; L\otimes\mathcal{L})^{-1} \simeq \omega_D^{\frac{1}{2}} \simeq (\omega^D)^{\frac{1}{2}} = \omega^{D/2}$

Θ_L

- algebraic construction of Θ function!

- Now can start from any abelian variety instead of E

$\Rightarrow \Theta_L^{(g)}$ modular form of wt $\frac{1}{2}D$ on moduli stack $\mathcal{M}_g$ of $(A, \mathcal{L})$ polarized abelian varieties.

Borcherds' congruence: $(\Theta_{L_1} - \Theta_{L_2}) \cdot x = 0 \quad \forall\, x \in Ext^i_{\mathcal{M}}(1, \omega^n)$ $i>0$

for two L_1, L_2 of same dimension.

Can ask if this remains true of higher theta functions (sums over sublattices rather than vectors!)

$\boxed{\text{Analogy}}$

Elliptic Curves		TMF
$\mathcal{O}$	$\longleftrightarrow$	$TMF^0(BS^1)$ (or really $TMF^0_{S^1}(pt)$ .... $TMF(BS^1) \simeq \mathcal{O}$ completed at identity)
$\mathcal{O}(-e)$	$\longleftrightarrow$	$\widetilde{TMF}^0(BS^1)$ reduced.
X $\downarrow$ $\mathcal{M}$	$\longleftrightarrow$	$TMF^0(X)$
$\mathcal{E}$ $\downarrow$ $\mathcal{M}$ elliptic curve	$\longleftrightarrow$	$X = BS^1$
Line bundle $\mathbb{L}(V)$ X $\downarrow$ $\mathcal{M}$	$\longleftrightarrow$	V = vector bundle on X "Thom ison" (nonconstant) $TMF^0(X^V)$
$E\otimes L$	$\longleftrightarrow$	$S^1\otimes L$

$\mathcal{L} \otimes L \longleftrightarrow ??$

- Note Witten genus $MO\langle 8\rangle \to TMF$ gives canonical orientation to $BO\langle 8\rangle$ bundles.

Have maps $BU\langle 6\rangle \to BO\langle 8\rangle$ ($c_1 = c_2 = 0$) — $()^3$ of any rep has $BU\langle 6\rangle$ structure

— now ignore $BO\langle 8\rangle$ mfs : $w_2 = p_1 = 0$

$$\begin{array}{ccccc} \mathbb{Z}[L^*] & \longrightarrow & & & \text{Rep of } S^1\otimes L \,/\, BO\langle 8\rangle \text{ mfs} \ni V \\ \cup & & & & \\ I & \longrightarrow & I/I^3 & \longrightarrow & I/I^3,\ 1-\sigma \cong (\mathrm{Sym}^2 L)^* \otimes \mathbb{Z}_{>} \end{array}$$

(I^3 lifts through $BU\langle 6\rangle$)

σ = complex conj (real theory)

-- take rep with $w_2 = 0$ $\frac{1}{2}p_1$ = canonical form

So $\mathcal{L} \otimes L \longleftrightarrow V$ constructed as above

- topological θ-fn would be maps $\Sigma^{D/2} BS^1 \otimes L^V \longrightarrow TMF$ $\leftarrow MO\langle 8\rangle$
- or really get E^2 term of SS converging to this...

Borcherds tells us this map in E^2 term survives to the limit, to give such a map

In equivariant bordism: find that theta manifold should arise geometrically as zero set of difference between two representations

TMF is really sheaf of spectra on $\mathcal{M}$ in étale topology lifting $\mathcal{O}$ (global sections) .. any étale map to $\mathcal{M}$ will give theory : invert level you get a map

"Should" have sheaf of spectra in flat topology on (Fgps) & $\mathcal{M} \to$ (Fgps) is flat $\Rightarrow$ should give sheaf on TMF for free... Global sections on (Fgps) should just be sphere spectrum.