Voevodsky
5/14/98

M. Hopkins - Steenrod Operations in Morava K-Theories

Milnor for schemes: $K_n^M \to H_M$ motivic $\simeq$ some étale coh.

Typical element in $K_n^M(k) \ni \{a_1, \dots, a_n\}$, $a_i \in k^\times$

when is this zero mod 2? want geometric answer..

$\{a\} \in K_1^M = k^*$, $\{a\} \equiv 0\ (2)$ iff a is a square

$\iff x^2 - ay^2$ has a solution!

Generalize via Pfister quadrics ..

Denote $\langle a \rangle = x^2 - ay^2$ quadratic form

$\langle\langle a_1, \dots a_n \rangle\rangle = \langle a_1 \rangle \otimes \dots \otimes \langle a_n \rangle$

Given $a_1, \dots a_n \Rightarrow$ Pfister quadric

$Q_{\bar a} = \{ \langle\langle a_1, \dots a_{n-1} \rangle\rangle = t^2 a_n \}$ Projective variety

dim $Q_{\bar a} = 2^{n-1} - 1$.

$Q_{\bar a}$ has a rational point over k iff $\{a_1, \dots a_n\} = 0$ in $K_n^M(k)/2$

Infinite join / Čech nerve of cover:

χ_a = geometric realization of simplicial object $\left| \begin{array}{c} \vdots \\ \downarrow\downarrow\downarrow \\ Q_{\bar a} \times Q_{\bar a} \\ \downarrow\downarrow \\ Q_{\bar a} \end{array} \right|$

- ~~iff~~ we have a point it's contractible

$\Rightarrow \chi_a \sim * \iff Q_a$ has a point.

- homotopy theoretic explanation.

- Related to classifying space of family of groups:

family of subgroups of Galois groups of fields where Q_a has a point ...

Contractible: $\chi_a \to$ Spec k is a weak equivalence.

Milnor conjecture pf is elaborate induction.. important part:

$H_M^{n+1,n}(\chi_a, \mathbb{Z}_{(2)}) = 0$

More useful to look at "mapping cone" of χ_a (which will be contractible..)

Define $\chi_a \underbrace{\to \text{Spec } k \to}_{\text{cofibration}} S^{1,0} \wedge \bar\chi_a$

So $S^{1,0} \wedge \bar\chi_a$ is mapping cone. - $\bar\chi_a$ fiber in stable category (shift indices in mapping cone.)

Really want

$*$ $H^{n+1,n}(\bar\chi_a, \mathbb{Z}_{(2)}) = 0$

Known : Rost : $H^{2^n-1,2^{n-1}}(\overline{\mathcal{X}}_a, \mathbb{Z}_{(2)}) = 0$

Induction : certain other groups are zero. Need : *.

Voevodsky constructs cohomology operations to connect these groups, then proves these are monomorphisms.

$\Rightarrow \quad H^{n+1,n} \hookrightarrow H^{2^n-1,2^{n-1}}(\overline{\mathcal{X}}_a) = 0.$

Bockstein

$$\begin{tikzcd} H^{**}(X,\mathbb{Z}) \arrow[r, "\times 2"] & H^{**}(X,\mathbb{Z}) \arrow[r] & H^{**}(X,\mathbb{Z}/2) \arrow[r] \arrow[dr, bend right, "\beta"'] & H^{*+1,*}(X,\mathbb{Z}) \arrow[d] \\ & & & H^{*+1,*}(X,\mathbb{Z}/2) \end{tikzcd}$$

E = another cohomology theory

$E^{-n,-m}(pt) \ni [\alpha : S^{n,m} \to E] \in E^0(S^{n,m}) = E^{-n,-m}(\operatorname{Spec} k)$

$$\begin{tikzcd} S^{n,m} \wedge E \arrow[r] \arrow[dr, "\alpha"'] & E \wedge E \arrow[d] \\ & E \end{tikzcd}$$

Think "$\alpha = 2$" - analog of multiplication by 2 maps.

$$S^{n,m} \wedge E \xrightarrow{\cdot\alpha} E \longrightarrow E/\alpha$$

$$(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2)$$

$$\begin{tikzcd} E^{*+n,*+m}(X) \arrow[r, "\cdot\alpha"] & E^{*,*}(X) \arrow[r] & E/\alpha^{*,*}(X) \arrow[r] \arrow[dr, "\beta_\alpha"'] & E^{*+n+1,*+m}(X) \arrow[d] \\ & & & E/\alpha^{*+n+1,*+m}(X) \end{tikzcd}$$

$\Rightarrow$ (the sequence above)

Bockstein as differential

$\beta^2 = 0 \ (\beta_\alpha^2 = 0) \Rightarrow$ B. s.s.

$$H^*(E/\alpha^{**}(X)\,;\beta_\alpha) \text{ "}\Longrightarrow\text{" } \check{\alpha}^{-1}E^{**}(X)$$

Ordinary cohomology $\mathbb{Z}/2 \Longrightarrow$ 2-adic cohomology.

$\overline{\mathcal{X}}_a$ is contractible in more situations :

$\mathcal{X}_\alpha \times V \to V$ is a homotopy equivalence if

$$\begin{tikzcd} Q_\alpha \times V \arrow[d] \\ V \arrow[u, bend right, dotted] \end{tikzcd}$$

has a section $\Rightarrow$ cofiber is contractible.

$$\mathcal{X}_\alpha \times V \longrightarrow V \longrightarrow S^{1,0} \wedge \overline{\mathcal{X}}_\alpha \wedge (V_+)$$

"half smash product" (mod out base point)

Obvious V to take : $V = Q_a$

$$\begin{tikzcd} Q_a \times Q_a \arrow[d] \\ Q_a \end{tikzcd}$$

$$S^{1,0} \wedge \overline{\mathcal{X}}_\alpha \wedge V_+ \longrightarrow S^{1,0} \wedge \overline{\mathcal{X}}_\alpha \wedge (\operatorname{Spec} k_+) = S^{1,0} \wedge \overline{\mathcal{X}}_\alpha$$

- anything in the pushforward of $S^{1,0} \wedge \overline{\mathcal{X}}_\alpha \wedge V_+$ is zero since it's contractible..

Doesn't work in motivic cohomology

$\to$ use motivic bordism.

Transfer

$$\begin{tikzcd} & 0 \arrow[d, equal] & \\ MGL^{**}(\bar{X}_a) \arrow[r] \arrow[rr, bend right, "\text{multiplication by } [V] \in MGL^{2d,d}(k)"'] & MGL^{**}(\bar{X}_a \wedge V_+) \arrow[r, "\text{push forward}"] & MGL^{**}(\bar{X}_a) \end{tikzcd}$$

Bockstein ss for V $\Rightarrow$ $H^*(MGL/[V]^{**}(\bar{X}_a) ; \beta_V) = 0$

- Bockstein is exact.

Assume V = a projective quadric, dim $2^d - 1$

$$\begin{tikzcd} MGL/[V] \arrow[r, "\text{Bockstein}"] \arrow[d, "\text{Thom reduction}"'] & S^{2^{d+1}-1,\, 2^d-1} \wedge MGL/[V] \arrow[d] \\ H\mathbb{Z}/2 \arrow[r, "\text{Milnor Operation}", "Q_d"'] & S^{2^{d+1}-1,\, 2^d-1} \wedge H\mathbb{Z}/2 \end{tikzcd}$$

ordinary cohomology

[analog of construction of Steenrod operations]

$\Rightarrow$ Q_d is exact.

Now Morava K-theories arise - as homotopy fiber of map on bottom; cohomology to which Q_d-Bockstein SS converges.

* Any subvariety of Pfister quadric maps back to it

$\longrightarrow$ intersect with generic hyperplanes.

$\longrightarrow$ find quadrics V of any dimension mapping to Pfister - dim $V = 2^d - 1$ for any $d \le n-1$.

So Q_d is exact on $H^{**}(\bar{X}_a)$ for any $d \le n-1$.

$$\begin{tikzcd} H^{n+1,n}(\bar{X}_a) \arrow[r, "Q_1"] & \cdot \arrow[r, "Q_2"] & \cdots\cdots \arrow[r, "Q_{n-2}"] & H^{2^n-1,2^{n-1}}(\bar{X}_a) = 0 \\ & H^{**}=0 \text{ induction} \arrow[u] & & \end{tikzcd}$$

Want each to be a monomorphism. By exactness, & induction - -