Great Lakes Conf
Northwestern
4/28/01

M. Hopkins – Twisted K-theory & Verlinde Algebra
(w/ D. Freed, C. Teleman)

Verlinde algebra: G compact Lie group & level $\lambda \in H^4(BG,\mathbb{Z})$

assume λ pos definite, $(\text{quad fns on } \mathfrak{h})^W = (\text{quadratic fns on } \mathfrak{g})^G = H^4(BG,\mathbb{R})$

Let $\Sigma = S^1 \times S^1$ – transgress λ: $\Sigma \times BG^\Sigma \longrightarrow BG$

$$H^4(\Sigma \times BG^\Sigma) \longleftarrow \lambda$$
$$\downarrow \int_\Sigma$$
$$H^2(BG^\Sigma,\mathbb{Z})$$

BG^Σ = space of maps $\mathrm{Map}(\Sigma, BG)$

BG^Σ = free loop space $\mathrm{Map}(S^1, \mathrm{Map}(S^1, BG))$
$$\cup$$
$$LG = \mathrm{Map}(S^1, G) \qquad G = L(BG) \text{ \underline{pointed maps}}$$

$\lambda \mapsto H^2(LG,\mathbb{Z}) \rightsquigarrow$ central extension $S^1 \to \widetilde{LG} \to LG$.

$BG^\Sigma = \mathrm{Bun}_G(\Sigma)$ – can rigidify by looking at G-bun with connection
$\Rightarrow$ line bundle $\mathcal{L}$ on $\mathcal{M}_G(\Sigma)$ = principal G-bundles with connection on Σ.
Choose complex structure on $\Sigma \Rightarrow \mathcal{L}$ gets hol structure.

Let V_λ = free abelian group on {irreps of $\widetilde{LG}$ ~~of level~~ + standard S^1 action}
– positive energy complex reps $\longrightarrow$ finite rank group.

$V_\lambda \otimes \mathbb{C} \simeq$ hol sections of $\mathcal{L}$ on $\mathcal{M}_G(\Sigma)$.
$G = S^1$ this is space of θ-fns: think of as <u>nonabelian thetas</u>.
V_λ also appears as conformal blocks in WZW model.

CFT $\Rightarrow$ V_λ is a <u>ring</u>, in fact a Frobenius algebra.
– the Verlinde algebra. Get formulas for nonabelian thetas for Σ of higher genus.

<u>Frobenius algebra</u>: algebra A/R + trace $tr: A \to R$
with $\langle x,y\rangle = Tr(xy)$ nondegenerate
Ex. • $\mathbb{Z}[F]$ F finite group
• $H^*(M)$ M manifold (Poincaré duality)
generally f.d. connected graded Hopf algebras

"<u>Theorem</u>" $V_\lambda \simeq K^G_{\zeta(\lambda)+\sigma}(G)$
twisted equivariant K-theory of $G \circlearrowleft G$ by conjugation ([illegible])

Note $G \times G \longrightarrow G$ is Ad-equivariant $\Rightarrow K^G_{\xi(\lambda)+0}(G)$ is a ring: Pontryagin product

$(\xi(\lambda)$ = twisting, 0 = dimension$)$

in fact gets Frobenius alg. structure.

Formula: G simply connected, simple

$V_\lambda = R[G]/I$ rep ring mod ideal

I = set of reps whose characters vanish on certain conjugacy classes.

- support of I disjoint from support of augmentation ideal $\Rightarrow$ can't have simple topological interp since BG completes rep ring.

Proved for G simply connected, checked for $SO(3)$, ...

Twisted K-theory: K-theory with coeffs in local system...

Suppose $M = \bigcup U_i$ space. To write fn on M write f_i on $U_i \longrightarrow \mathbb{C}$, $f_i = f_j$ on $U_i \cap U_j$

Twisted function: give multipliers $\lambda_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}^*$

$\rightsquigarrow f_i = \lambda_{ij} f_j$ twisted function (section of flat line bundle)

Cocycle condition: $\{\lambda_{ij}\} \in H^1(M, \mathbb{C}^*) = H^1(M, GL_1(\mathbb{C}))$

K-theory: vector bundles on opens agreeing on overlap — now twist by unit on overlaps.

"$f \in K(U_i)$, $\lambda_{ij} \in K(U_i \cap U_j)^*$"

Define $GL_1(K)$ space with $[M, GL_1(K)] = K^0(M)^*$

(value 1 on basepoint)

$GL_1(K) = \{\pm 1\} \times K(\mathbb{Z}, 2) \times BSU = (\{\pm1\} \times BU)_\otimes$

Obvious units in K theory: line bundles L

same for $-L$ $\Rightarrow \pm 1 \times K(\mathbb{Z}, 2)$

$\Rightarrow$ group of "graded line bundles" over M.

A twisting of K-theory is a cocycle representing an element of $H^1(M; GL_1(K))$

Theorem (Segal, May) $GL_1(K)$ is an infinite loop space, so $H^0(M, GL_1(K)) := [M, GL_1(K)]$ is a homology theory.

So $H^1(M, GL_1(K))$ is first cohomology group with these coeffs

$\rightsquigarrow$ can define $H^i(M; GL_1(K))$ $\quad i \geq 0$

All cohomology groups split $H^i(M; GL_1(K)) = E_1^i(M) \times E_2^i(M)$

$\leftrightarrow \{\pm 1\} \times K(\mathbb{Z},2) \quad \times \quad BSU.$

$E_1^0(M) = H^0(M;\mathbb{Z}/2) \times H^2(M;\mathbb{Z})$ graded lines

$E_1^1(M) = H^1(M;\mathbb{Z}/2) \times H^3(M;\mathbb{Z})$ with
funny group structure : $(a,b)+(c,d) = (a+c,\ \beta(ac)+b+d)$
β Bockstein giving nontrivial extension.

$$\begin{tikzcd} H^4(M;\mathbb{Z}) \arrow[r, hook] & E_1^2(M) \arrow[r] & H^2(M,\mathbb{Z}/2) \arrow[r, "\beta"] & H^5(M,\mathbb{Z}) \end{tikzcd}$$

- normalized K-invariant.

$E_1^1(M)$ twist K-theory - give local systems in K-theory

Other factor : $E_2^i(M) = bsu^i(M)$ after completing at any prime
(connective K-theory). Will only consider twistings of K-theory
by elts of $H^1(M,\mathbb{Z}/2)\times H^3(M;\mathbb{Z}) = E_1^1(M)$

Given $\xi \in E_1^1(M) \Rightarrow$ twisted K-homology, $K_{\xi+n}(M)$: version of $K_n(M)$ twisted by ξ
& $K^{\xi+n}(M)$ [Donovan Karoubi]

Products : $K_{\xi+n}(S)\otimes K_{\eta+m}(T) \longrightarrow K_{\xi+\eta+n+m}(S\times T)$:

$$\begin{tikzcd} K_{\xi+0}(G)\otimes K_{\xi+0}(G) \arrow[r] & K_{\xi\otimes 1+1\otimes\xi+0}(G\times G) \arrow[d, "?"] \\ & K_{\xi+0}(G) \end{tikzcd}$$

- need $G\times G \longrightarrow G$

(*) $\xi\otimes 1+1\otimes\xi \leftarrow \xi$: need ξ primitive to make this a ring

(*) is guaranteed if ξ is transgressed from an elmt of $H^2(BG, GL_1(K))$
(automatically primitive)

$$\begin{tikzcd} H^4(BG,\mathbb{Z}) \arrow[r, phantom, "\subset"] & H^2(BG, GL_1(K)) = E_1^2(BG) \arrow[d] \\ & H^2(BG,\mathbb{Z}/2) \end{tikzcd}$$

<u>Example</u> $G = SO(3)$

$H^3(SO(3),\mathbb{Z}) = \mathbb{Z}$ but only primitives $\xi\in H^3$ are
ones congruent to 0 mod 2.

- can get all elements of all an $H^2(BG, \mathbb{Z}/2)$ part!
$(1,1) \in H^1(\mathbb{Z}_2) \times H^3(\mathbb{Z})$ <u>is</u> primitive.

Classifying space B^2 for $E_1^2(X, -) = [X, B^2]$
turns out to be Postnikov section of BSO - first 2 nontrivial π_i's.

$\Rightarrow$ $\exists$ canonical elt $\kappa \in H^2(BSO; GL_1 K)$, in fact in $E_1^2(BSO)$

G any connected Lie group $\Rightarrow h \in E_1^2(BG)$,

$$BG \xrightarrow{\text{adjoint}} BSO(\dim G) \hookrightarrow BSO$$
$$E_1^2(BG) \ni h \longleftarrow \kappa$$

$h \leftrightarrow$ dual Coxeter number. (Casimir eigenvalue of adjoint)

Def $\tau(\lambda) = -$ transgression of $\lambda + h$

$\lambda + h \in E^2(BG) \underset{\text{transgression}}{\rightsquigarrow} E^1(\text{loop space of } BG) = E^1(EG \times_G G)$ adjoint

$\Rightarrow$ can make $\lambda + h$ land in $H^1_G(G; GL_1(\mathbb{C}_G))$

$$V_\lambda = K^G_{-\tau(\lambda+h)+0}(G)$$

Actually RHS gives Frobenius algebra for any nondegen λ, &
G not nec. simply connected.
Adjoint rep not spin $\Rightarrow$ must include $\mathbb{Z}/2$ part
of the invariant

Get map $K^G_{\tau(\lambda)}(1) \longrightarrow K^G_{\tau(\lambda)}(G)$
$\qquad \| $
$K^G_G(pt) = R(G)$ gives the map $R(G)/I \simeq V_\lambda$.

Critical level $k + h^\vee = 0$
$\Rightarrow$ $K^G_0(G)$ identified with Kähler diff of rep ring.

Atiyah-Segal, Mathai: model for twisted K-theory by $H^3(X, \mathbb{Z}) \ni H$

$PU \simeq K(\mathbb{Z}, 2)$ $H^3(X, \mathbb{Z}) \rightsquigarrow PU$ bundle over M

$PU \to E$
$\quad \downarrow$
$\quad M$

Fredholm bundle over M. homotopy classes of sections