I. Grojnowski : Satake for Double Loop Groups 10/27/04

Theorem (w/ C. Teleman) Let $\hat{\mathcal{B}}$ be the "double loop Grassmannian". Then there is a category of perverse sheaves $\mathcal{P}$ on $\hat{\mathcal{B}}$ & a functor to vector spaces ("deRham cohomology") $\mathbb{H} : \mathcal{P} \to \mathrm{Vect}$ s.t.

- For $\mathcal{A} \in \mathcal{P}$, $\mathbb{H}(\mathcal{A})$ is an integrable representation of ${}^L\hat{\mathfrak{g}}$

Moreover there are two convolution structures on $\mathcal{P}$
- First: finite & braided $\xrightarrow{\mathbb{H}}$ fusion tensor product
- Second: not finite but symmetric $\xrightarrow{\mathbb{H}}$ usual tensor product

To prove:
- need guide to phenomena, so not to get lost
- technology for homotopy theory in alg. geometry, for very big spaces
- some hard local geometry

1. Lusztig's Satake isomorphism (after S. Kato, R. Brylinski)
(work on dual side here: switch $G, G^\vee$).

$${}^L\hat{Gr} = {}^LG((z)) / {}^LG[[z]]$$

Fiebig-G. Teleman: Strong Macdonald conjecture ⋮

i Orbits of ${}^LG[[z]]$ $\longleftrightarrow$ X^+ dominant weights

closure relation: $\mu \leq \lambda \iff \lambda - \mu \in \bigoplus_{\alpha \in \Phi^+} \mathbb{N}\alpha$

(ii) Kazhdan-Lusztig property: pointwise pure. $\Rightarrow$ local IC Poincaré polys

$$K^{IC}_{\mu\lambda} = \sum (-1)^i \dim \mathcal{H}^i \, i_\mu^* IC_\lambda \, t^{i/2} \quad \in \mathbb{Z}[t, t^{-1}]$$

(iii) Define Hall-Littlewood polynomial

$$P_\lambda = \frac{1}{W_\lambda(t)} \sum_{w \in W} w\Big(e^\lambda \prod_{\alpha \in \Phi^+} \frac{1 - te^{-\alpha}}{1 - e^{-\alpha}}\Big)$$

$W_\lambda(t)$ = Poincaré polynomial of stabilizer group

χ_λ Weyl character

$$\chi_\lambda = \sum_{\mu \leq \lambda} K_{\mu\lambda} P_\mu \quad \text{change of basis}$$

where $K_{\mu\lambda} = K^{IC}_{\mu\lambda}$ J: poly nomials

- $K_{\mu\lambda} \neq 0 \implies \mu \leq \lambda$, $K_{\mu\mu} = 1$
  If $\mu < \lambda$, $K_{\mu\lambda}(0) = 0$
- $K_{\mu\lambda} \in \mathbb{N}[t]$

$P_\lambda(1) = \sum_{w \in W\lambda} e^{\lambda}$. $K_{\mu\lambda}(1) = \dim L(\lambda)_\mu$

"monomial symmetric function"

$K_{\mu\lambda}$ is t-analog of weight multiplicities

Examples 1. $P_0 = 1$

2. $K_{0\theta} = \sum t^{d_i}$ adjoint representation.

2. Combinatorics : $\mathfrak{g}$ Kac-Moody Lie algebra (symmetrizable)
$\leadsto (\mathfrak{h}, X, \Pi, \Pi^\vee)$, $\Pi = \{\alpha_i\} \in \mathfrak{h}^*$, $\Pi^\vee = \{h_i\} \subseteq \mathfrak{h}$ X lattice in $\mathfrak{h}^*$ containing $\sum \mathbb{Z}\alpha_i$

$\dim \mathfrak{h} - \# I = \# I - rk(\text{Cartan matrix})$

t-analog of Weyl denominator $\Delta_t = \prod_{\alpha \in \Phi^+} (1 - t e^{-\alpha})^{\dim \mathfrak{g}_\alpha}$

$\Delta = \Delta_1$ X^+ dominant weights _ X^0 imaginary roots

Definition Hall-Littlewood polynomial
$$P_\lambda = W_\lambda(t)^{-1} \sum_{w \in W} w \left(e^\lambda \frac{\Delta_t}{\Delta} \right) \quad \text{(as formal power series)}$$

Proposition I. $\exists$ $a_{\mu\lambda} \in \mathbb{Z}[t]$ polynomials,
s.t. $P_\lambda = \sum_{\substack{\mu \leq \lambda \\ \text{dominant}}} a_{\mu\lambda} \chi_\mu$, $a_{\lambda\lambda} = 1$

where $\chi_\lambda = P_\lambda|_{t=0}$ Weyl character of integrable rep.

$P_\lambda \in \mathrm{Rep}\, G$.

e.g. P_0 : if $\mathfrak{g}$ is fin dim
$\mu \in X^+$, $\mu \leq 0 \Rightarrow \mu = 0$

So prop. 1 $\Rightarrow P_0 = 1$

If $\mathfrak{g} = \mathfrak{sl}_2[z, z^{-1}] + \mathbb{C}c + \mathbb{C}d$, $\underset{\in \mathfrak{h}}{\delta} = \alpha_0 + \alpha_1$ imaginary root

$\mathbb{Z}$-torsion of dimensions for double loop group: infinitely many dominant weights less than a given one, due to imaginary weights

<u>Proposition</u> $P_0 = 1 \iff \mathfrak{g}$ has no imaginary roots

Example : $\mathfrak{gl}_\infty$

$$\sum_{j \geq 0} \prod_{i \neq j} \frac{1 - t\, z_i / z_j}{1 - z_i / z_j} = \frac{1}{1-t}$$

<u>Def</u> $K_{\mu\lambda}$ is the inverse matrix, i.e.

$$\chi_\lambda = \sum_{\substack{\mu \leq \lambda \\ \mu \in X^+}} K_{\mu\lambda} P_\mu$$

Hope $K_{\mu\lambda}$ local intersection cohomology on $\overline{\mathcal{B}}$

A <u>KL polynomial</u> is a polynomial satisfying Kazhdan–Lusztig combinatorics, recursion procedure.

Completed character ring :

$$R[X]^\wedge = \varprojlim_k R[X] \big/ R[\alpha \in \Phi^+ : ht\, \alpha \geq k]$$

KL involution $^+$: $\mathbb{Z}[t, t^{-1}][X]^\wedge \circlearrowleft$

$( )^+ : e^\lambda \longmapsto e^\lambda \qquad t \longmapsto t^{-1}$

$$\Rightarrow \chi^\mu \longmapsto \chi^\mu, \quad P_\lambda \longmapsto P_\lambda + \sum_{\substack{\mu < \lambda \\ \mu \in X^+}} \mathbb{Z}[t, t^{-1}] P_\mu$$

<u>Proposition</u> $K_{\mu\lambda} \in \mathbb{Z}[t]$ such that
i. $K_{\lambda\lambda} = 1$ ii $K_{\mu\lambda}|_{t=0} = 0$ if $\mu < \lambda$
iii $\deg K_{\mu\lambda} = ht(\lambda - \mu)$

... gives notion of relative change of strata

<u>Corollary</u> $K_{\mu\lambda}$ are the unique polynomials satisfying i) & ii) s.t $\left(\sum K_{\mu\lambda} P_\lambda\right)^+ =$ itself.

$K_{\mu\lambda}$ satisfy KL recursion relations (so are probably local IC poincaré polynomials)

<u>4. Examples</u> $\mathfrak{g} = \bar{\mathfrak{g}}((z)) + \mathbb{C}c + \mathbb{C}d$

$q = e^{-\delta}$ $(a)_\infty = \prod_{n \geq 0} (1 - q^n a)$

<u>Theorem</u>. 1. $P_0 = \prod_1^\ell \frac{(qt^{d_i})_\infty}{(qt^{d_i - 1})_\infty}$

$d_1, \dots d_\ell$
$\shortparallel$
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2. $\bar{\mathfrak{g}}$ simply laced $\Lambda_0 \longleftrightarrow$ basic rep

$$P_{\Lambda_0} = \left(\prod_1^\ell \frac{1}{(qt^{d_i})_\infty}\right) \chi_{\Lambda_0}$$

<u>Case of $\mathfrak{sl}_2$</u> : P's are infinite sums (of finite products of rational fns), & this is equating them with infinite products

$$\sum_{n \in \mathbb{Z}} \frac{(t^{-1}w)_{2n}}{(tqw)_{2n}} \frac{1 - qw^{2n}}{1 - w^{-1}} t^{2n} = \frac{(qt^2)_\infty}{(qt)_\infty}$$

Ramanujan ${}_1\psi_1$ sum

ii gives Baily ${}_4\psi_4$ sum.

Lemma P_0 identity is equivalent to Macdonald's Constant term conjecture ... Cherednik proved using DAHA ...

$(\lambda=0)$

$w \in W_{aff}$

$$\frac{\Delta_+}{\Delta}$$ look for terms involving only q's $(t=0)$,

Dimensions of strata: $ht(\theta)=\theta$ maximal root $= \max(d_i)$

$$ht\,\delta - ht\,\theta + 1 = \max\{d_i\}$$

Theorem $K_{\mu\lambda} \in \mathbb{N}[t]$ if λ not imaginary $\in X^0$

$\mathcal{B}$ flags for your Kac-Moody

Proposition $P_\lambda(t)$ is the character of

$$\sum_{i,j} (-1)^i t^j H^i(\mathcal{B}, \Omega^j \otimes \mathcal{O}_\lambda)$$ $\longrightarrow$ integrable highest weight rep

Δ_+

$\shortparallel$

$\prod_{\alpha \in \Phi^+} (1-te^{-\alpha})^{\dim \mathfrak{g}_\alpha}$

Euler char of cohomology for fixed j, make generating Sum.
... fixed point formula ... Weyl character formula
(Cousin complex calculations)

Let $\tilde{E}_\mu = H^*(T^*\mathcal{B}, \mathcal{O}_\mu)$ symmetric algebra

$$\approx \prod_{\alpha>0} (1-te^{\alpha})^{\mathrm{mult}\,\alpha}$$: not integrable representation

.. can't filter here by line bundles unlike finite dim case.

Theorem $H^p(T^*\mathcal{B}, \mathcal{O}_\lambda) = 0$ if $p>0$, $\lambda \in X^+$

$\mathcal{B}$ formally smooth

$$H^0(T^*\mathcal{B}, \mathcal{O}) = \left(\mathrm{Sym}\,\mathfrak{g}((z)) \underset{Z^+}{\otimes} \mathbb{C}\right) \otimes Z^+$$

$$Z^+ = (\mathrm{Sym}\,\mathfrak{g}[[z]])_{\mathfrak{g}[[z]]}$$
$$= \mathrm{Sym}((\mathfrak{g}/\!/G)[[z]])$$

Finite dim: ~~$\emptyset$~~ this is Grauert–Riemenschneider, use Hodge theory of flag variety to deduce

$$T^*\mathcal{B} \xrightarrow{\mu} \mathcal{N} \qquad R^{>0}_{f_*} \mathcal{O}_\lambda = 0 \quad \ldots \lambda \text{ any}$$

Here Hodge–deRham classical collapse at E_1.

<u>Double loops</u>

G reductive, take $\mathbb{G}_m$ bundle $\hat{G}^{der}$ over $G((s))$ central extension

$$\hat{G} = \hat{G}^{der} \rtimes \mathbb{C}^*$$

$$\hat{\mathcal{B}} := \hat{G}^{der}[[z]] \backslash \hat{G}((z)) / \hat{G}^{der}[[z]] \qquad \text{quotient stack}$$

$$\pi_0 \hat{\mathcal{B}} \twoheadrightarrow \pi_0(\mathbb{C}^*((z))) = \mathbb{Z}$$

$$\hat{\mathcal{B}}_c = \text{components lying over } c \in \mathbb{Z}$$

Orbits: ~~all flag orbits have~~ σ.

$$W \ltimes X \backslash W \ltimes \mathbb{Z} \times X \times X / W \ltimes X = X \times \mathbb{Z} / W \ltimes X$$

But have ρ-shift: this parametrizes highest weight modules at level $c - h^\vee$.

So $\quad G((s))[[z]] \backslash G((s))((z)) / G((s))[[z]]$ is studying critical level ...,

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$$(g(z), 1)\,(f(z), q)\,(g(z), 1)^{-1} = g(qz) f(z) g(z)^{-1}$$

On c component $q = z^c$

action of lattice is shifted by c.

X formally smooth, $\hat{X}$ completion of $X \hookrightarrow X \times X$

$\mathcal{D}$-mod on $X \Longleftrightarrow \mathcal{O}$-module on stack $\hat{X} \rightrightarrows X \times X$

(not formally smooth $\Rightarrow$ take stratifying site).

Stack $\in$ model category of simplicial sheaves with Illusie weak equivalence