M. Kapranov – Higher dimensional Class Field Theory 5/11/00

I. Aspects of the usual CFT.

1. Standard geometric formulation: $X/\mathbb{F}_q$ smooth projective curve

$Y \downarrow$ unramified abelian cover

$X \ni x \rightsquigarrow Fr_x \in Gal(Y/X)$

Reciprocity: If $f \in \mathbb{F}_q(X)$, $\operatorname{div} f = \sum n_i x_i$, then

$\prod Fr_{x_i}^{n_i} = 1$

$\Rightarrow$ homomorphism $\operatorname{Pic} X \to Gal(Y/X)$.

$0 \to \underset{\text{finite}}{\operatorname{Pic}^0(X)} \to \operatorname{Pic} X \xrightarrow{\deg} \mathbb{Z} \to 0$ ($\mathbb{F}_q$-points)

Theorem (1) $\operatorname{Pic}^0(X) = Gal(X^{ab}_{unr}/X)$ max ab unramified, not extending field of constants

(2) $\operatorname{Pic} X = W^{ab} = \pi_1^{ab} \twoheadrightarrow \pi_1^{ab}(\mathbb{F}_q) = \hat{\mathbb{Z}}$ Weil group

((1) not quite canonical: need splitting of 2 -- canonically look at $\bar{X}/\bar{\mathbb{F}}_q$ & construct its cover... (1) needs a choice of an $\mathbb{F}_q$-point of X).

If $Y \to X$ ramified at $D \subset X$ (ramification subscheme)

$\Rightarrow Fr_x$ $x \notin |D|$.

$f \equiv 1 \pmod D \Rightarrow \prod Fr_{x_i}^{n_i} = 1$ $\operatorname{div} f = \sum n_i x_i$

$\operatorname{Pic}_D(X)$ relative Picard group (quotient out by f as above only)

$\downarrow$ – line bundles trivialized on D

$Gal(Y/X)$.

$$\varprojlim_D \mathcal{O}^*/k(X)^* \to W^{ab}(k(X))$$

2. Self-duality of Jacobian:

k any field, X curve$/k$, $\underline{\operatorname{Pic}}^0(X)$ abelian variety

Classical: it is self dual, $\cong$ line bundles of deg 0 on itself.

Better: Picard stack $\operatorname{Pic}(X)$ (any degree) with $\otimes$

Then this stack is self-dual! isomorphic to tensor-maps $\operatorname{Hom}_{\otimes}(\operatorname{Pic} X, \mathbb{G}_m\text{-Tors})$

multiplicative line bundles

---- $\mathbb{Z}$ of degree is dual to $\mathbb{G}_m$ of automorphisms on Pic^0 ---

Deligne pairing $\quad \mathrm{Pic}\,X \times \mathrm{Pic}\,X \longrightarrow \mathbb{G}_m\text{-Tors}$

$\qquad L \,,\, M \longmapsto \langle L, M\rangle$ (1-dim vector space)

Construction for L, M very ample:

$H^0(L) \otimes H^0(M) \supset \nabla = \{(s,t) \mid s,t \text{ have common zero}\}$ irred. hypersurface

$\Rightarrow \nabla$ has irreducible equation – a homogeneous polynomial (resultant)

$R = R(s,t) \quad : \; R(s,t) = 0 \iff \exists$ common zero.

- R is defined only up to scalar

$\iff$ $\exists$ canonical 1-dim vector space $\langle L, M\rangle$ where canonical R takes values

(True for any irred hypersurface in a linear space: collection of level sets of equations )

Deligne notation $\langle s,t\rangle$ for $R(s,t)$ – by multiplicativity.

Meaning $\quad \begin{matrix}\mathfrak{X} \\ \downarrow \\ S\end{matrix}$ family of curves, L, M line bundles on $\mathfrak{X}$

$\Rightarrow \langle L, M\rangle$ line bundle on S.

$$c_1(\langle L, M\rangle) = \int_{\mathfrak{X}/S} c_1(L)\, c_1(M)$$

3. Local version: Contou-Carrère self-duality of $\mathbb{G}_m((t))$

-- group indscheme. $\quad \mathbb{G}_m((t)) \times \mathbb{G}_m((t)) \longrightarrow \mathbb{G}_m$

$\qquad f \,,\, g \longmapsto \exp\{\oint \log f \; d\log g\} \quad$ f, g Λ-pts

(note resultant ~ Steinberg symbol!

Resultant of $f, 1-f = 1$ no common zero)

For Λ a field obtain tame symbol

$$\{f, g\} = (-1)^{\operatorname{ord} f \operatorname{ord} g} \, f^{\operatorname{ord} g} / g^{\operatorname{ord} f}\,(0)$$

[Beilinson $\quad x \in X \quad U = X \setminus x$: have $\mathrm{Pic}_c(U)$ compact support – true group scheme (triv near x) $\quad \mathcal{O}^* \to \mathrm{Pic}_c U \to \mathrm{Pic}\,X$

$\mathrm{Pic}\,U = \mathrm{Pic}\,X / \mathrm{Pic}_x X \quad \leftarrow$ line bundles triv outside x.

"core indcted category stuff"

Then $\mathrm{Pic}\,U$ & $\mathrm{Pic}_c U$ are dual

$$\left.\begin{matrix} k[[t]]^* \longrightarrow \mathrm{Pic}_c U \longrightarrow \mathrm{Pic}\,X \\ \mathrm{Pic}_x X \longrightarrow \mathrm{Pic}\,X \longrightarrow \mathrm{Pic}\,U \end{matrix}\right\} \text{dual exact sequences.}$$

$\mathrm{Pic}_x X = k((t))^* / k[[t]]^*$

]

4. Poincaré duality for $G = \mathrm{Gal}(\bar{K}/K)$ K local nonarchimedean field

$H^2(G, \mu_\infty) = \mathbb{Q}/\mathbb{Z}$

M finite G-module $\Rightarrow$ perfect pairing given by cup product

$H^i(G, M) \times H^{2-i}(G, \mathrm{Hom}(M, \mu_\infty)) \longrightarrow \mathbb{Q}/\mathbb{Z}$

5. Local CFT : K local as before

$W^{ab}(K) = K^*$ Weil group

Poincaré duality explanation : $m \in \mathbb{Z}$, Kummer

$W^{ab}/mW^{ab} = H_1(\mathrm{Gal}, \mathbb{Z}/m) \overset{P.D.}{=} H^1(\mathrm{Gal}, \mu_m) = K^*/(K^*)^m$

II Aspects of the higher-dimensional CFT

1. "Standard" formulation : $X/\mathbb{F}_q$ smooth projective, $\dim X = n$.

Y unramified Abelian $\rightsquigarrow$ Fr_x as before

$\downarrow$

X For every curve C on X $\rightsquigarrow$ reciprocity law

$\Rightarrow$ homomorphism $CH_0(X) \longrightarrow \mathrm{Gal}(Y/X)$ 0-cycles

(reciprocity law, relations given by curves)

<u>Theorem</u> $CH_0(X) \xrightarrow{\sim} W^{ab}(X)$ (unramified part of π_1)

$\quad\quad\quad\quad \|$

monodromy $\to Fr_x$

$H^n(X, \underline{K_n})$: analog of $\mathrm{Pic} = H^1(X, \mathcal{O}^*)$

$x \circlearrowleft \mathrm{spec}\,\mathbb{F}_q$ X $\quad$ $\underline{K_n}$ = sheaf of $U \mapsto K_n(\mathcal{O}(U))$

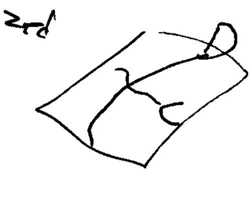
Y ramified : try to form some $CH_0(X, D)$

$\downarrow$ - relations come from generalized

$X \supset D$ (hypersurface) Jacobians of C mod $C \cap D$

(limit statement on Galois group holds,

estimating conductors is hard ...)



2. "Self $(n+1)$-ality" of $\mathrm{Pic}\, X$ $\quad X/k$ $\dim = n$

$\mathrm{Pic}\, X \times \mathrm{Pic}\, X \times \cdots \times \mathrm{Pic}\, X \longrightarrow \mathbb{G}_m\text{-Tors}$

$L_0 \quad L_1 \quad \ldots \quad L_n \longmapsto \langle L_0, \ldots L_n \rangle$

via resultant of n sections

$\mathfrak{X}$ family of n-dim varieties $\quad \langle L_0, \ldots L_n \rangle \in \mathrm{Pic}\, S$

$\downarrow$ $\quad c_1 \langle L_0, \ldots L_n \rangle = \int_{\mathfrak{X}/S} \prod c_1(L_i)$

S

Bloch : To $CH_0(X)^0$ <u>deg</u> 0 part have biextension that gives the duality ...

$(\mathrm{Pic}\, X)^n \longrightarrow CH_0$ .. would like to replace

$\mathrm{Pic}\, X \times \mathrm{Pic}\, X^n \longrightarrow \mathbb{G}_m$ tors by pairing Pic & CH_0 ...

This pairing will kill Albanese kernel..

$2\frac{1}{2}$. Local fields and adeles in n dimensions (Parshin, Beilinson)

Say $n=2$, X surface$/k$. $C \subset X$ irr. curve

$\Rightarrow$ C-adic valuation on $k(X) \supset \mathcal{O}_{X,C} \supset m_C$

$\Rightarrow$ completion $k(X)^\wedge_C$ complete discrete valued field with residue field $k(C)$ -- still "global".

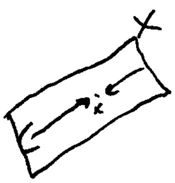
$$k(X)^\wedge_C = \varinjlim_{i \in \mathbb{Z}} \varprojlim_{j \geq i} m_C^i / m_C^j$$

$x \in C \rightsquigarrow$ completion $k(C)^\wedge_x$ local field

<u>Main observation</u>: $\exists$ natural "enhancement" of $k(X)^\wedge_C$ which is a complete D.V. field $k(X)^\wedge_{x,C}$ with residue field $k(C)^\wedge_x$

<u>Construction</u> $\mathcal{M}_C \subset \mathcal{O}_X$ sheaf of ideals of C

$j_2 : X - C \hookrightarrow X$ $\quad j_1 : C - \{x\} \hookrightarrow X$



$\forall i \in \mathbb{Z} \Rightarrow \mathcal{M}_C^i \subset j_{2*} j_2^* \mathcal{O}_X$ coherent

$m_C^i / m_C^j = \Gamma(C_{gen}, \underbrace{\mathcal{M}_C^i / \mathcal{M}_C^j}) $ sections on generic point of C
$\longrightarrow$ coherent sheaf on X, supported on C.

$$k(X)^\wedge_C = \varinjlim_i \varprojlim_j \Gamma(C_{gen}, \mathcal{M}_C^i / \mathcal{M}_C^j)$$

$$= \varinjlim_{\substack{\mathcal{J} \subset j_{2*} j_2^* \mathcal{O}_X \\ \text{coherent}}} \varprojlim_{\substack{\mathcal{J}' \subset \mathcal{J} \\ \mathrm{supp}(\mathcal{J}/\mathcal{J}') \subset C}} \Gamma(C_{gen}, \mathcal{J}/\mathcal{J}')$$

Now if $\mathcal{F} \in \mathrm{Coh}\, X$ is supported on C,
can first complete it at x

$$\mathcal{F}^\wedge_x = \varprojlim_i \mathcal{F}/\mathcal{M}_x^i \mathcal{F} = \varprojlim_{\substack{\mathcal{F}' \subset \mathcal{F} \\ \mathrm{Supp}\, \mathcal{F}/\mathcal{F}' \subset \{x\}}} \mathcal{F}/\mathcal{F}'$$ (restriction to $\hat{X}_c$.

Further $\exists$ restriction to <u>punctured</u> formal nbhd of x in C

$$\mathcal{F}^{\wedge\circ}_x = \varinjlim_{\substack{\mathcal{H} \subset j_{1*} j_1^* \mathcal{F} \\ \text{coherent}}} \varprojlim_{\substack{\mathcal{H}' \subset \mathcal{H} \\ \mathrm{supp}\, \mathcal{H}/\mathcal{H}' \subset \{x\}}} \mathcal{H}/\mathcal{H}'$$

So finally $k(X)_{x,C} = \varinjlim_{\substack{\mathcal{G} \subset j_{2*} i_2^* \mathcal{O}_X \\ \text{coh}}} \varprojlim_{\substack{\mathcal{G}' \subset \mathcal{G} \\ \operatorname{Supp} \mathcal{G}/\mathcal{G}' \subset C}} \varinjlim_{\substack{\mathcal{H} \subset j_{1*} i_1^* \mathcal{G}/\mathcal{G}' \\ \text{coh}}} \varprojlim_{\substack{\mathcal{H}' \subset \mathcal{H} \\ \operatorname{Supp} \mathcal{H}/\mathcal{H}' = x}} \mathcal{H}/\mathcal{H}'$

j_2^* means <u>sheaf</u>-theoretic preimage not Grothendieck...

<u>General picture</u> ($\dim X = n$ arbitrary)

For any flag of irreducible subvarieties $(Y_1 \subset \dots \subset Y_m) = Y_\bullet$

$\dim Y_i = d_i$ (not nec full flag)

$\Rightarrow$ "completion" $K_{Y_1 \subset \dots \subset Y_m}$ of $\mathcal{O}_X$, a ring (not nec field) s.t. ($\mathcal{O}_X$-algebra)

1) $K_{\{x\}} = \hat{\mathcal{O}}_{X,x}$ (x 0-dim point)

2) $K_X = k(X)$

[$n = 2$ $\quad k(X)^\wedge_{x,C} = K_{x,C,X}$]

3) For a complete flag of smooth subvarieties $K_{Y_1 \dots Y_n}$ is a field, complete wrt a discrete valuation, with residue field $K_{Y_1 \dots Y_{n-1}}$ for Y_{n-1} ($Y_n = X$)

4) If a flag $Z_\bullet$ refines flag $Y_\bullet$ $\Rightarrow$ embedding $K_{Y_\bullet} \subset K_{Z_\bullet}$.

5) $\forall$ sequence of integers $0 \le d_1 < \dots < d_m \le n$

$\exists$ natural restricted product $\mathcal{A}_{d_1 \dots d_m} \subset \prod_{\substack{Y_1 \subset \dots \subset Y_m \\ \dim Y_i = d_i}} K_{Y_1 \dots Y_m}$

- defined directly as iterated limit for coherent sheaves on X.

$\Longrightarrow$ adélic complex

$$\mathcal{A}^\bullet = \Big\{ \underset{0}{\oplus \mathcal{A}_i} \longrightarrow \underset{1}{\bigoplus_{i<j} \mathcal{A}_{ij}} \to \dots \to \underset{n}{\mathcal{A}_{123\dots n}} \Big\}$$

cosimplicial $\mathcal{O}_X$-algebra.

6) For $\forall$ quasicoherent sheaf F on X, $\mathcal{A}^\bullet \otimes_{\mathcal{O}_X} F$ calculates $H^i(X, F)$

<u>Examples</u> a) X curve

$$\begin{matrix} \mathcal{A}_0 = \hat{\mathcal{O}} & \\ \oplus & \longrightarrow \mathcal{A}_{01} = \text{standard adeles } \mathcal{A}_X \\ \mathcal{A}_1 = k(X) & \end{matrix}$$

automorphic theory: $G(\mathcal{A}_1) \backslash G(\mathcal{A}_{0,1}) / G(\mathcal{A}_0)$ $\quad G$ a group..

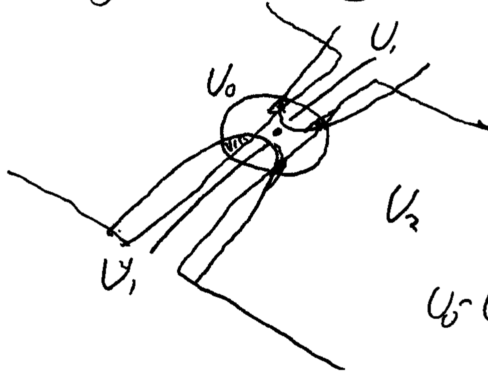
b) X surface:

$$\begin{matrix} \text{compact} & \prod_x \hat{\mathcal{O}}_{X,x} = & \mathcal{A}_0 & & \mathcal{A}_{01} & & \\ & & \oplus & & \oplus & & \\ \prod^{\text{res}}_{C \subset X} & \operatorname{Fun}(\text{formal neigh of } C^{\text{gen}}) = & \mathcal{A}_1 & \longrightarrow & \mathcal{A}_{02} & \longrightarrow & \mathcal{A}_{012} \\ & & \oplus & & \oplus & & \\ \text{discrete} & k(X) = & \mathcal{A}_2 & & \mathcal{A}_{12} & & \end{matrix}$$

$\mathcal{A}_{ij} = \mathcal{A}_i \otimes_{\mathcal{O}} \mathcal{A}_j$

- Čech covering for cover : all formal discs, generic point of X & formal nbhd of all generic pts of curves.

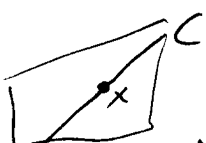
Picture for just one flag fixed



U_0 = form nbhd of x in X
U_1 = " " of $C \setminus x$ in X
$U_2 = X \setminus C$
$U_{ij} = U_i \cap U_j$ etc

$U_0 \cap U_1 \cap U_2 \sim$ torus . . .

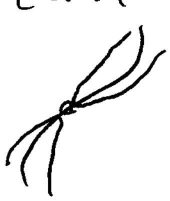
[Beilinson f mero function on variety. What is $f(x)$, $x \in X$?
On curve: either number, or infinity...
On surface: can't assign value to any mero fn.
e.g. $f = x/y$ on $\mathbb{C}^2$ - can't assign value to point zero! need curve passing through 0, then get angle of curve



— so need a flag to assign value!
Procedure: f has pole on C $\Longrightarrow$ ∞
not $\Longrightarrow$ evaluate $f|_C$ at x.
- indep of choice.

Class field theory : if covering is ramified ~~over~~ a point need more data than just a point.. plain nbhds of points can be very complicated, branching hard -> local data hard.

But fix flag  - if have ramification along curve : look in tiny nbhd of the curve cone going to zero exponentially fast so no other curves hit this nbhd



$\Longrightarrow$ very simple topology, just 2-dim tors - most local situation, simple & can't be made smaller
$\sim$ spectrum of 2-d loc field.]

Def An n-dim local field is a complete DVF whose residue field is an $n-1$ dim local field, & 0-dim local field = finite field.

1-dim: $\mathbb{Q}_p, \mathbb{F}_q((t))$ [loc compact]

2-dim: $\mathbb{Q}_p((t)), \mathbb{F}_q((t_1))((t_2))$, another type (Laurent series in t with p-convergent tail ...)

2 & higher don't have structure of topological ring! – multiplication not continuous... but are ring objects in $(\text{Ind-Pro})^n$(finite sets)

4. Pairing of K-groups of an n-dim local field

n-dim local $E = E_n$

$$\begin{array}{l} E = E_n \\ \cup \\ \mathcal{O}_E \longrightarrow E_{n-1} \\ \qquad\qquad \cup \\ \qquad\qquad \mathcal{O}_{E_{n-1}} \longrightarrow \cdots \longrightarrow E_0 = \mathbb{F}_q \end{array} \implies \tilde{\mathcal{O}}_E = \text{full preimage of } \mathbb{F}_q, \quad \tilde{\mathcal{O}}_E \xrightarrow{R} \mathbb{F}_q$$

e.g. in $\mathbb{F}_q((t_1))\ldots((t_n))$

Taylor series in t_n whose zeroth coeff is Taylor in t_{n-1} whose zeroth coeff is ...

$\Gamma = E^*/\tilde{\mathcal{O}}_E^* \simeq \mathbb{Z}^n$ not canonically.

Multivaluation: $\text{ord}: E^* \longrightarrow \Gamma$.

Γ has a canonical filtration, quotients canonically $\mathbb{Z}$

$\Rightarrow \Lambda^n\Gamma \simeq \mathbb{Z}$. $\Rightarrow$ have determinant of n vectors in $\Gamma \in \mathbb{Z}$, sd

Analog of tame symbol $E^* \times \cdots \times E^* \longrightarrow \mathbb{F}_q^*$

iterated tame symbol $(f_0, \ldots, f_n) \longmapsto \{f_0, \ldots, f_n\}$

$$= (-1)^c \, R\left(\prod_{i=0}^n f_i^{(-1)^i \det(\text{ord} f_0, \ldots \widehat{\text{ord} f_i} \ldots \text{ord} f_n)}\right)$$

Sign c = "det" of an $n \times (n+1)$ matrix over $\mathbb{F}_2$ of $(\text{ord}(f_i))$ mod 2. (Khovanskii)

For a field F, Milnor K group $K_\cdot^M(F) = \Lambda^\cdot(F^*)/_{x \wedge (1-x) = 0}$

$F \supset \mathcal{O} \to k$ gives boundary map

$\partial: K_i^M(F) \longrightarrow K_{i-1}^M(k)$

& $\{f_0, \ldots f_n\}$ is composition $K_{n+1}^M E_n \xrightarrow{\partial} K_n^M E_{n-1} \longrightarrow \cdots \longrightarrow K_1^M E_0 = \mathbb{F}_q^*$

$\Leftrightarrow$ as in pairing on Pic's, can write

as $E^* \otimes K_n^M(E) \longrightarrow \mathbb{F}_q^*$

"dual" to E^*

5. n-dim <u>Contou-Carrère symbol</u> ??

$$\mathbb{G}_m((t_1))\cdots((t_n)) \times \cdots \times \mathbb{G}_m((t_1))\cdots((t_n)) \longrightarrow \mathbb{G}_m$$

should exist & provide local version for $\langle L_0, \ldots L_n \rangle$. ← group object in $(\mathrm{Ind\,Pro})^n$(schemes of fin type)

6. <u>Poincaré duality</u> for $G = Gal(\bar{E}/E)$ E n-dim loc

$$H^{n+1}(G, \mu_m^{\otimes n}) = \mathbb{Q}/\mathbb{Z} \quad \text{-- prime to } p \ldots$$

& for any finite G-module M have perfect pairing by cup product

$$H^i(G, M) \times H^{n+1-i}(G, Hom(M, \mu_\infty)) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

7. <u>Local n-dim CFT</u>: $W^{ab}(E)$ = some quotient of $K_n^M(E)$

$n=2$: K_2^M can be made into a topological group, & W^{ab} = max. Hausdorff quotient.

$n>2$ more subtle. If $E \subset F$ finite abelian extension,

$$Gal(F/E) = K_n^M(E) / Norm(K_n^M F)$$

Can explain via Poincaré duality: Bloch-Kato thm.

$$W^{ab}/m = H^1(Gal, \mathbb{Z}/m) = H^n(Gal, \mu_m^{\otimes n}) \xleftarrow{\sim} K_n(E)/m$$

Local-global for $n=2$: $X/\mathbb{F}_q$ surface

$$W^{ab}(X) = CH_0(X) = H^2(X, \underline{K}_2) \overset{\text{adelic complex}}{=\!=} \frac{K_2(\mathcal{O}_{012})}{K_2(\mathcal{O}_{01}) + K_2(\mathcal{O}_{02}) + K_2(\mathcal{O}_{12})}$$

- only possible in abelian case! don't have enough sides in non-abelian groups!

$$W^{ab}(k(X)) = \frac{K_2(\mathcal{O}_{012})}{K_2(\mathcal{O}_{02}) + K(\mathcal{O}_{12})} \quad \text{(only things related to 2)}.$$

<u>Note</u> $H^n(X, \underline{K}_n) = H^n(X, \underline{K}_n^M) = CH_0(X)$