U of Chicago
2D Langlands
10/28/04

M. Kapranov - Higher Langlands ?

1. Higher dim class field theory

Local

n-dim local fields

$\mathbb{F}_q((t_1))\dots((t_n))$

$\mathbb{Q}_p((t_1)),\dots((t_{n-1}))$

$\mathbb{R}$

Global

absolute ($/\mathbb{Z}$) n-dim schemes $X^n/\mathbb{F}_q$

Weil group $W(K) \hookrightarrow Gal(\bar{K}/K)$

$$W^{ab} \simeq K^{top}_{n,\,Milnor}(K) \overset{\text{“}N\geq n\text{”}}{\hookrightarrow} \overset{\text{for}}{H_n}(GL_N(K))$$

(taking profinite completion of K-theory) topological : need to take into account topology of fields, take completed tensor product as symbols

So 1-dim reps of $W \longleftrightarrow$ continuous Steinberg symbols

$c(x_1,\dots,x_n)$, $x_i\in K^*$

↑

come from n-cocycles of $GL_N(K)$

Simplest representations : $W \longrightarrow \mathbb{Z}\{Fr\} \longrightarrow \mathbb{C}^*$

$1 \longmapsto s$

$K=\mathbb{F}_q((t_1))\dots((t_n))$: tame symbol

$\{x_1,\dots x_n\}\in K_0(\mathbb{F}_q)=\mathbb{Z} \overset{s}{\hookrightarrow} \mathbb{C}^*$

$n \longmapsto s^n$

$\longleftrightarrow$ determinantal cocycle

$\underline{n=2}$: $K=L((t))$ ← ordinary local field

$\{x_1,x_2\}\in L^*$, extends to canonical element in $H^2(GL_n(K),L^*)$: Tate (determinantal) central extension.

$H^2(GL_n(K),L^*)$ ~~etc~~ : take $L^*\xrightarrow{|z|^s}\mathbb{C}^*$

↓

$H^2(GL_nK,\mathbb{C}^*)$

Tame symbol is $\partial: K_2(K)\longrightarrow K_1(L)$

$\partial\{x,y\}=\{x,y\}_{tame}$

This is the symbol corresponding to standard 1-dim rep of W

$\gamma_s \in H^2(GL_n K, \mathbb{C}^*) \longleftrightarrow$ s-character of W

In this example everything comes from cohomology.

<u>Another</u> (more fundamental?) <u>formulation</u>

K n-dim local field: $\Gamma = \mathrm{Gal}(\bar{K}/K)$ satisfies Poincaré duality in dimension $n+1$: $\exists$ canonical element

$\xi \in H^{n+1}(\Gamma, \mu_\ell^{\otimes n}) = \mathbb{Z}/\ell \Rightarrow$ perfect pairing

$H^i(\Gamma, \mu_\ell^{\otimes j}) \Leftrightarrow H^{n+1-i}(\mu_\ell^{\otimes n-j})$

<u>$n=1$</u>: $H^1(\mathrm{Gal}\, K, \mu_\ell) \otimes H^1(\mathrm{Gal}\, K, \mathbb{Z}/\ell) \to \mathbb{Z}/\ell$

(ordinary local field) $\quad \| \quad K^\times/(K^\times)^\ell$ by Kummer map

true for any field.

Langlands for 1-dim local field:

r-dim reps of $W \longleftrightarrow$ some reps of $GL_r K$

$H^1(W, GL_r \mathbb{C})$ $\qquad$ "$H^1(GL_r K, $"$GL_\infty$"$)$"

when considering $\mathbb{C}$-representations $\qquad$ ∞-dim reps

n-dim case:

Reps of W is always an H^1... so it should be put in correspondence with some nonabelian H^n...

$n=2$: nonabelian H^2

Γ a group, A an abelian group $\Rightarrow$

$H^2(\Gamma, A)$ = actions of Γ on categories: A-gerbes

A-gerbes: category $\mathcal{C}$ with $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ A-torsor
$\mathrm{Hom}(x,y) \times \mathrm{Hom}(y,z) \longrightarrow \mathrm{Hom}(x,z)$ A-maps

e.g. $A = \mathbb{C}^*$ $H^2(\Gamma, \mathbb{C}^*)$ = actions of Γ on category $\mathrm{Vect}_{\mathbb{C}}$, or even 1-dim $\mathrm{Vect}_{\mathbb{C}}$:

$g \in \Gamma$ $\varphi_g : \mathcal{C} \to \mathcal{C}$ $\varphi_{gh} \overset{c(g,h)}{\Longrightarrow} \varphi_g \circ \varphi_h$
+ 2-cocycle condition for triples g, h, k.

"Candidate" for nonabelian H^2 of Γ: actions of Γ on categories

Classical concept of character of Γ: $\chi(gh) = \chi(g)\chi(h)$, $\chi(g) \in \mathbb{C}^*$

generalizations

2-cocycles
$\xi(g,h) \in \mathbb{C}^*$

matrix representations
$\rho(g) \in GL_n$

?
common generalization
is nonabelian 2-cocycles?

[K_n is part of H_n, things we actually "see" come from homology, eg tame symbols etc.: $K_{\mathbb{Q}} \Leftrightarrow$ primitive elements in homology]

Analog of a character of a representation for action on category (w. Nora Ganter)

Suppose have action on category: $\forall g$ $\varphi(g) : \mathcal{C} \to \mathcal{C}$

$\mathrm{Tr}(\text{functor } A : \mathcal{C} \to \mathcal{C}) = ?$

<u>Def</u> $\mathrm{Tr}(A)$ = natural transformations$(1_{\mathcal{C}}, A)$

Ex. $\mathcal{C} = D^b(\mathrm{Coh}\, X)$. $K \in D^b(X \times X)$ kernel

$A(F) = Rp_{2*}(p_1^* F \overset{L}{\otimes} K)$

$\Rightarrow$ $\mathrm{Tr}\, A = R\Gamma(X, K|^{L}_{\Delta})$

If Γ acts on $\mathcal{C}$: $\forall g \Rightarrow \mathrm{Tr}\ \rho(g)$ "sheaf" on Γ which is conjugation equivariant ... character sheaf (Lusztig character sheaves should arise as traces in this sense for action on $D^b(G/B)$ --- constructible sheaves)

If $g, h \in \Gamma$ commuting $\Rightarrow$ 2-character

$$\chi_{\mathcal{C}}^{(2)}(g,h) = \mathrm{tr}(g \mid \mathrm{Tr}\ \rho(h))$$

function on pairs of commuting elements, invariant under simultaneous conjugation. — 2-class functions

-- appear in elliptic cohomology:

$\mathrm{Ell}(BT) =$ 2-class functions (Hopkins-Kuhn-Ravenel)

$\mathrm{Ell}(\Sigma) \sim$ certain families of categories over Σ, like $K(\Sigma) \sim$ vector bundles over Σ.

So Nonabelian $H^2 \sim$ actions on categories

Naive idea of matrix representations: direct sums of $\mathrm{Vect}_{\mathbb{C}}$

-- modules over ring category $(\mathrm{Vect}_{\mathbb{C}}, \oplus, \otimes)$

e.g. $\mathrm{Vect}_{\mathbb{C}}^{\oplus n} = \mathrm{Coh}(\underbrace{\bullet\ \bullet\ \bullet \ldots \bullet}_{n})$

$\Rightarrow A = \| A_{ij} \|$ matrix of vector spaces

Problem! few invertible matrices (since dim vector spaces ≥ 0)

Note • Usual characters = class fns which are elementary projectors under convolution

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2. Hecke operators

X curve $/\mathbb{F}_q$ $\quad x \in X \quad \mathrm{Bun}_r(X)$

$\Rightarrow$ ~~function~~ operator $T_{x,i} : \mathbb{C}[\mathrm{Bun}_r(X)] \circlearrowleft$

$$T_{x,i}(f) = \sum [E' : 0 \to E' \to E \to k_x^{\oplus i} \to 0]\ f(E')$$

sum over such modifications

X n-dim variety $/\mathbb{F}_q$. Construct operators on $\mathbb{C}[\mathrm{Bun}_r X]$

E' as above would no longer be a bundle, rather torsion-free sheaf $\Rightarrow$ better perhaps to modify along divisors.

Let $\mathrm{Coh}_m(X)$: Purely m-dimensional coherent sheaves

i.e. $\mathcal{F}$ s.t. $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) = 0$ for $i \neq n-m$

$\forall\, \mathcal{F} \in \mathrm{Coh}_m$, $\mathcal{E} \in \mathrm{Coh}_{m+1}$ can consider modifications

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{F} \to 0$$

$$\Rightarrow \mathcal{E}' \in \mathrm{Coh}_{m+1}$$

$\Rightarrow \exists$ operators $T_{\mathcal{F}}$ on $\mathbb{C}[\mathrm{Coh}_{m+1}(X)]$

e.g. punctual Hecke operators act on sheaves with 1-dim support etc.

Satisfy Hall algebra relations:

$$T_{\mathcal{F}'} \circ T_{\mathcal{F}''} = \sum_{\mathcal{F}} c^{\mathcal{F}}_{\mathcal{F}'\mathcal{F}''} T_{\mathcal{F}}$$

$$c^{\mathcal{F}}_{\mathcal{F}'\mathcal{F}''} = \#\{\Phi \subset \mathcal{F} : \Phi \simeq \mathcal{F}', \mathcal{F}/\Phi \simeq \mathcal{F}''\}$$

Hall algebra of Coh_m.

On a curve $T_{x,i} \longleftrightarrow \Lambda^i(Fr_x)$ Frobenius in Galois gp

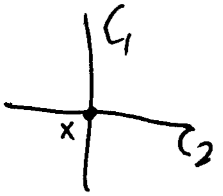
X surface : Coh_0 supported at pts, its Hall alg acts on Coh_1 = 1-dim support, whose Hall algebra acts on $\mathrm{Coh}_2 = \mathrm{Bun}(X)$

$\mathrm{Hall}(\mathrm{Coh}_1) \curvearrowright \mathbb{C}[\mathrm{Bun}_r X]$: Wildly behaved action.

For noninteresting curves operators commute

<u>Conjecture</u> $\forall$ point $x \in X$ should have a class in

$HH^2(\mathrm{Hall}(\mathrm{Coh}_1)) \longleftrightarrow$ Frobenius of pts in Galois group



Think of point as giving relations between curve operators .. x gives a nontrivial relation in HH^2..

Such relations become Serre relations in case of ADE graphs... in general don't know structure

So points don't <u>act</u>, but give some such higher cohomological objects, corresponding to $Frob_x$.

--- look for this as basis of relation

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3. "Generalization" of elliptic modules

X surface $/\mathbb{F}_q$ $\qquad A = \mathbb{F}_q[X - D]$

$\cup$

D ample divisor $\qquad$ (think in 1-dim of this embedding in rings of diff op, or Frob-linear polynomials)

K = completion of $\mathbb{F}_q(X)$ along D:
dvfield, with residue field $= \mathbb{F}_q(D)$:
ie semilocal field.

$A \subset K$ discrete.

Drinfeld exponential $\quad e_A(z) = z \prod_{a \in A \setminus 0} \left(1 - \frac{z}{a}\right)$

q-power series

(ie of form $z + c_1 z^q + c_2 z^{q^2} + \dots$)

$e_A(nz) = P_n(e_A(z)) \quad$ for $n \in A$

P_n is a q-power series $= nu + \dots u^q + \dots$

— get $\boxed{\text{formal module}}$ not elliptic module,

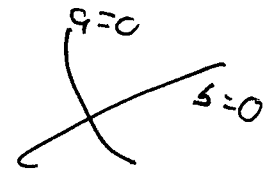
$P_{n+m} = P_n + P_m, \quad P_{nm} = P_n(P_m) = P_m(P_n)$

Case of curves: P_n's are finite degree polynomials, have finiteness, have moduli spaces

Finiteness properties $\quad K/A \xrightarrow{e_A} K$ as abelian groups

$\to$ get nontrivial A-module structure on K/A.

Let $a, b \in A$ ... i.e. a, b give curves and $\{a=b=0\}$ a 0-dim subscheme



$\to$ form Koszul complex $0 \to K \longrightarrow K^{\oplus 2} \longrightarrow K \to 0$

$$x \longmapsto (P_A(x), P_B(x))$$

$$(u,v) \longmapsto P_b(u) - P_a(v)$$

exact away from middle term, where cohomology is a finite abelian group --- analog of torsion of an elliptic module -- follows from injectivity of K as A-module: $a * x = P_a(x)$. $(K, *) \simeq K/A$

K injective $/A \Rightarrow$ complex calculates $\mathrm{Ext}^\bullet(A/(a,b), A)$

(we're considering A as the trivial A-lattice in K)

$H^1 \simeq \boxed{\mathrm{Ext}^2 \simeq A/(a,b) \otimes \Omega^2_A}$

More generally for $L \subset K$ locally free A-submodule write e_L, $P_n^L(u)$

Koszul: $|H^1| = (\mathrm{rk}\, L) \cdot A/(a,b)$

or more canonically middle cohomology is $M/(a,b) \otimes_A \Omega^2$

Need global resultant: $K \xrightarrow{\varphi} K^2$, $\quad x \longmapsto u = P_a(x)$, $v = P_b(x)$

$\mathrm{Im}(\varphi) \subset K^2$ has unique up to constant analytic [illegible] $R_{a,b}(u,v)$

Expect $P_b(u) - P_a(v)$ to be a polynomial of $R_{a,b}(u,v)$

4. Eisenstein series for Kac-Moody groups

Usual geometric Eisenstein series: Consider maps $X \xrightarrow{f} G/B$

$\deg f \in$ ~~[illegible]~~ $H_2(G/B) = L$ coweight lattice

Map_d finite dimensional.

$$E(z) = \sum_{d \in L} |Map_d|_{\mathbb{F}_q} \cdot z^d \qquad z \in T^\vee = Hom(L, \mathbb{C}^*)$$

... this series has support more or less in dominant cone, & gives a rational function of z satisfying functional equation wrt Weyl group W.

Could replace $|Map_d|_{\mathbb{F}_q}$ # points by motive, or topological Euler characteristic, Hodge polynomial etc - anything additive wrt cut & paste (ie "measure")

ρ-shifted W-action
$$E(wz) = \prod_{\substack{\alpha>0 \\ w(\alpha)<0}} \zeta(z^\alpha)/\zeta(qz^\alpha) \cdot E(z)$$

Now $G \rightsquigarrow \hat{G}$ Kac-Moody group

$$1 \to \mathbb{C}^* \to \hat{G} \to G(\mathbb{C}((t))) \to 1$$

determinantal central extension from GL_∞ for GL_n case

- from Sato Grassmannian

Drinfeld Poor result for construction of $\hat{G}$ compared with strength of people involved (Faltings, Deligne, Brylinski, ...)

- all fail for $G = E_8$ eg in family of curves acquiring singularities

Max torus of $(\mathbb{C}^* \ltimes \hat{G} = \tilde{G})$ is $\underset{z}{T} \times \underset{q}{\mathbb{C}^*} \times \underset{u}{\mathbb{C}^*}$

q: modular variable for elliptic curves

$\hat{W} = W \ltimes L$ acts

let $\begin{matrix} E \\ \downarrow \\ \mathcal{D} = \{|q|<1\} \end{matrix}$ universal elliptic curve

$$\begin{matrix} (4) \\ \downarrow \\ (E \otimes_{\mathbb{Z}} L)/W \end{matrix}$$ theta line bundle

then $\hat{T}_+/\hat{W}$ = total space of (4).

$\hat{T}_+ = \{|q|<1\} \subset \hat{T}$ - relation between characters of Kac-Moody groups & theta functions

"S-duality": X projective surface $/\mathbb{C}$

$Bun_G(X,n)$: semistable bundles with $c_2 = n$

$$F_G(q) = \sum \chi(Bun_G(X,n))\, q^n$$

should exhibit modular behavior, for a congruence subgroup

More general generating functions:

$Z \subset X$ curve

$Bun_{G,B}(X,Z,n,d)$: G-bundles on X, $c_2 = n$, with B-reduction along Z of degree $d \in L$

(degree of B-reduction of G-bundle on $Z \hookrightarrow Z \to G/B$)

$z \in T^\vee$: $E_G(q,z) = \sum_{n,d} \mu(Bun_{G,B}(X,Z,n,d))\, q^n z^d$

should have elliptic behavior in z, modular behavior in q.

Change of setup Fix a bundle P_0 on $X \setminus Z$

$$M_{G,P_0}(n) := \{(P,\tau) : \underset{c_2(P)=n}{P \text{ } G\text{-bundle on } X}, \ \tau : P|_{X \setminus Z} \to P^0\}$$

$M_{G,P_0}(n,d) = \{(P,\tau)$ as above + parabolic structure of degree $d\}$

Claim If $Z\cdot Z<0 \Rightarrow$ these spaces are finite dimensional & empty for $n<<0$

Evidence: $T_{[P,\sigma]} M_{G,P^0}(n) = H^1_Z(X, \mathrm{ad}\, P)$

$= H^0(Z, \underline{\mathcal{H}}^1_Z(X, \mathrm{ad}\, P))$ has filtration with quotients $\mathrm{ad}\, P|_Z \otimes N_{Z/X}^{\otimes i}$

(normal bundles).. have no sections for $i>>0$

Relation to maps into affine Grassmannian $Gr = G((t))/G[[t]]$

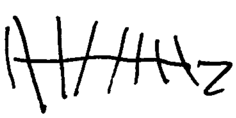
Say $X = \underset{Z\cdot Z=0}{Z} \times \mathbb{A}^1$. A G-bundle on $X = Z\times\mathbb{A}^1$ with a triv on $Z\times\{\mathbb{A}^1\setminus 0\}$

is a map $Z \longrightarrow G$.

In general have twisted situation: bundle of Grassmannians

finite dim version: $P \xrightarrow{G} Z \Rightarrow \mathrm{Flags}(P) \to Z$ twisted G/B bundles, we're considering sections

(ie $Bun_B \to Bun_G$: reductions of G-bundle to B)



$\hat{G}$-bundles $/Z \Leftrightarrow$ line bundle L over Z
+ principal bundle on $\mathrm{Tot}(L)\setminus 0$
+ determinental data for central extension
--- ruled surface case. More generally should consider G bundles on tubes around surface

Assume $\delta = Z\cdot Z<0$. write $\hat{G} = \tilde{G}\times\mathbb{C}^*$, $\hat{T} = T\times\mathbb{C}^*_q\times\mathbb{C}^*_v$

Write $\sum_{n,d} \mu(M_{G,P^0,B}(n,d))\, q^n z^d v^{-\delta}$

$\hat{E}(q,z,v)$ formal function on $\hat{T}$.

<u>Theorem</u> $E(q,z,v)$ extends to a meromorphic section of Θ^d on $(E\otimes_{\mathbb{Z}} L)/W \to D$

Pf: reduction to simple reflections

$$\tilde{E}(q,z,v) = \underbrace{E(q,z,v)}_{\xi} \cdot \prod_{\substack{\alpha>0 \\ \text{affine pos roots}}} \zeta(\xi^{\alpha}) / \zeta(\xi^{\alpha}\cdot \mathbb{L})$$

($\mathbb{L} = \mu(\mathbb{A}^1)$ motive) is W^{aff} invariant

<u>Example</u> $Z = \mathbb{P}^1 \subset X$ = ruled surface, $Z\cdot Z = -d$
($X \supset |\mathcal{O}_Z(-d)|$)

<u>Claim</u> G-bundles on $X \setminus Z \longleftrightarrow$ ~~ch~~ integrable characters of level d for $G^\vee$.
corresponding Eisenstein series are <u>characters</u>.
-- Kac-Moody bundles / $\mathbb{P}^1$ come from ~~ka~~ torus

For $P^0 \longleftrightarrow \pi$ irrep of affine of $G^\vee$
Then E_{P^0} is $\mathbb{L}$-deformation of the character
($\mathbb{L} = \mu(\mathbb{A}^1)$ = # points of field)

-- Hall polynomials