M. Kapranov - Hecke algebra & harmonic analysis over a 2-D local field 3/8/01

Hierarchy of Hecke algebras assoc to a simple alg group G

- Finite Hecke alg H_q $q \in \mathbb{C}$,

generators T_α α simple root

braid relations & $(T_\alpha + 1)(T_\alpha - q) = 0$

⇝ Basis T_w $w \in W$ $T_\alpha = T_{s_\alpha}$ s_α simple reflection

- Affine Hecke algebra $\hat{H}_q$ - same def as, as one for G simply connected, basis T_w $w \in \hat{W} = W \ltimes L$

L coweight lattice. $X_a = T_a$ $a \in L$ commute & form $\mathbb{C}[L]$.

G not simply connected: still basis T_w $w \in \hat{W} = W \ltimes L$

but $\hat{W}$ no longer gen. by simple reflections...

$(\pi_1(G) = 0)$ • Double affine Hecke algebra $\mathcal{H}_q$ (Cherednik)

Partial basis T_w $w \in \hat{W}$ as before - in particular X_a, $a \in L$

& elements Y_b $b \in L^\vee$ forming $\mathbb{C}[L^\vee]$ and

central element ξ (analogous to loop group central extension)

& relations: Y_b together with T_w for $w \in W$ finite form another copy of $\hat{H}_q$ in $\mathcal{H}_q$.

- e.g. $T_\alpha Y_b T_\alpha^{-1} = Y_b Y_\alpha^{-1}$ for α simple root

$T_0 Y_b T_0^{-1} = Y_b Y_\theta^{-1} \xi$ θ = max pos root

$\mathcal{H}_q$ contains Hecke algebras for two Langlands dual groups, joined over finite W

- comes from π_1 of configuration space of universal elliptic curve (A_n case).

Hierarchy of local fields (Parshin)

Finite	Local nonarch	2-dim local field
$\mathbb{F}_q$	$k = \mathbb{Q}_p$, $\mathbb{F}_q((t))$	$K = \mathbb{Q}_p((t))$, $\mathbb{F}_q((t_1))((t_2))$

$H_q \sim G(\mathbb{F}_q)$ $\hat{H}_q \sim G(k)$ as Hecke algebras in group theoretic sense

Recall Γ loc compact top. group $\supset \Delta$ compact subgroup

$\mathcal{F}_0(\Gamma)$ = cont. functions with compact support

algebra wrt convolution

Hecke algebra $H(\Gamma, \Delta) \subset \mathcal{F}_0(\Gamma)$ subalgebra of Δ-biinvariant functions

$\sim \mathcal{F}_0(\Delta \backslash \Gamma / \Delta)$

$H(\Gamma,\Delta)$ acts in $\mathcal{F}_{all}(\Gamma/\Delta)$ all fns, by Γ-inv operators.

$\Delta\backslash\Gamma/\Delta = \Gamma\backslash(\Gamma/\Delta \times \Gamma/\Delta)$ Γ orbits

$C \longmapsto \Sigma_C \subset \Gamma/\Delta \times \Gamma/\Delta$ orbit

If Δ is <u>open</u> then $\Delta\backslash\Gamma/\Delta$ discrete, and C (i.e. characteristic function $\mathbb{1}_C$) acts by the "Hecke operator" $T_C : \mathcal{F}(\Gamma/\Delta) \circlearrowleft$

$$(T_C f)(x) = \sum_{y \in \Gamma_C(x)} f(y) \qquad \Sigma_C(x) = \{ y : (x,y) \in \Sigma_C \}$$

– finite sum.

<u>Finite case</u>: We get $H_q = H(G(\mathbb{F}_q), B(\mathbb{F}_q))$ $B < G$ Borel

$B\backslash G/B = W$ Σ_w = Schubert correspondence $\longleftrightarrow T_w \in H_q$.

<u>p-adic case</u>: k local, res. field $\mathbb{F}_q$ $k \supset \mathcal{O}_k \xrightarrow{\pi_k} \mathbb{F}_q$.

$G(k)$ has 2 Borel-type subgroups:

- D_0 = Iwahori = $\{ g \in G(\mathcal{O}_k) \mid \pi_k(g) \in B(\mathbb{F}_q) \}$
- $D_1 = T(\mathcal{O}_k) \cdot N(k)$ – "connected component" of $B(k)$, neither compact nor open.

$D_i\backslash G(k)/D_j \simeq \hat{W}$ for any combination of i,j (Bruhat-Tits, Iwasawa)

$\hat{H}_q = H(G(k), D_0)$

<u>2-dimensional case</u> $K \supset \mathcal{O}_K \xrightarrow{\pi_K} k \supset \mathcal{O}_k \xrightarrow{\pi_k} \mathbb{F}_q$

$G(K)$ has 3 Borel-type subgroups

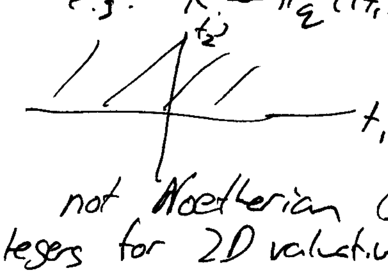
$\Delta_0 = \{ g \in G(\mathcal{O}_K) \mid \pi_K(g) \in G(\mathcal{O}_k),\ \pi_k\pi_K(g) \in B(\mathbb{F}_q) \}$
$= \pi_K^{-1}(D_0)$

$\Delta_1 = \pi_K^{-1}(D_1)$

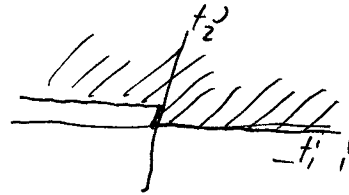
$\Delta_2 = T(\mathcal{O}_K')N(K)$

$\mathcal{O}_K' = \pi_K^{-1}(\mathcal{O}_k) \subset \mathcal{O}_K$: e.g. $K = \mathbb{F}_q((t_1))((t_2))$

$\mathcal{O}_K = \mathbb{F}_q((t_1))[[t_2]]$



$\mathcal{O}_K'$



not Noetherian (not $\mathbb{F}_q[[t_1]][[t_2]]$)

– i integers for 2D valuation on K.

Rank two lattice $\mathcal{E} = K^*/(\mathcal{O}_K')^*$: .

Powers $t_1^i t_2^j$ - not canonically $\mathbb{Z}^2$ - can change t_2 by power of t_1 ... but has natural sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \to & \mathcal{E} & \to & \mathbb{Z} & \to & 0 \\ & & k^*/\mathcal{O}_k^* & & & & \text{"}K^*/\mathcal{O}_k^*\text{"} & & \end{array}$$

$\Delta_i \backslash G(K)/\Delta_j \simeq \hat{W} = W \ltimes (L \otimes \mathcal{E})$ for any i,j ("Bruhat")

$L \otimes \mathcal{E} = T(K)/T(\mathcal{O}_k')$

Form "loop group central extension" - this is like p-adic loop group!

$1 \to k^* \to \widetilde{G(K)} \to G(K) \to 1$

given by tame symbol $K_2(K) \to k^*$ (Matsumoto)

& minimal Weyl scalar product $\psi : L \to L^\vee$

- think of $K = k((t))$ ⇝ usual loop group central extension.

· This is trivial over any of the Δ_i

($i = 0,1$ b/c symbol is trivial over $\mathcal{O}_K$, $i=2$: Matsumoto construction trivial on upper triang matrices)

$\tilde{\Delta}_i = \Delta_i \times \mathcal{O}_k^* \subset \widetilde{G(K)}$

$\tilde{\Delta}_i \backslash \widetilde{G(K)}/\tilde{\Delta}_j \simeq \tilde{W} = W \ltimes (\widetilde{L \otimes \mathcal{E}})$ → Heisenberg extension

by $\mathbb{Z}$ of $L \otimes \mathcal{E}$ corresponding to symplectic form

$\psi \otimes \sigma$ on $L \otimes \mathcal{E}$, $\sigma : \Lambda^2 \mathcal{E} \to \mathbb{Z}$

comes from short exact for $\mathcal{E}$ or from tame symbol.

"Hecke algebra" for $\tilde{\Delta}_i$ should be related to Cherednik $\mathcal{H}_q$.

⇝ e.g. wrt. $H(\text{center} = k^*, \mathcal{O}_k^*) \simeq \mathbb{C}[\mathfrak{z}, \mathfrak{z}^{-1}] \subset \mathcal{H}_q$

Basis vectors approximately match: double cosets ⟷ basis in $\mathcal{H}_q$.

- but double cosets have two copies of L, whereas $\mathcal{H}_q$ has $L, L^\vee$ ---- expect to get only part of $\mathcal{H}_q$ corresponding to $\psi : L \hookrightarrow L^\vee$

- Call this $\mathring{\mathcal{H}}_q$

<u>Natural idea</u>: try to make sense of $H(\widetilde{G(K)}, \tilde{\Delta}_i)$ some i & relate to $\mathcal{H}_q$ ...

<u>BUT</u> Not only $G(K)$ not locally compact, not even clear what topology it has - not clear for K itself!

Aim: Make sense of $H(\widetilde{G(K)}, \widetilde{\Delta_1})$ & relate to $\mathcal{H}_q$.

Earlier: Ginzburg-K tried to make sense of $H(\widetilde{G(K)}, G(\mathcal{O}_K))$ & got stuck at some point — which will come to the forefront here.

For rest of talk: ignore central extension (don't lose much in understanding).

Idea: try to define integral operators on <u>functions</u> on $G(K)/\Delta_1$ $= \hat{\Xi}$ ... but have to specify which functions!

What is $\hat{\Xi}$? $G(K) \supset I_K = \pi_K^{-1}(B(k))$

Iwahori: corresp to local field structure of K

$\Delta_1 \subset I_K \twoheadrightarrow L = T(k)/T(\mathcal{O}_k)$

so $\hat{\Xi} = G(K)/\Delta_1 \xrightarrow{L\text{-torsor}} G(K)/I_K =: \hat{F}$ "affine flag variety"

Assume $K = k((t)) \Rightarrow \hat{F} =$ affine flags over k
$= \varinjlim$ proj algebraic varieties $/k$.

In general still have $\hat{F} = \varinjlim$ compact profinite spaces.

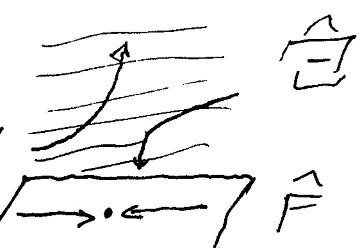
$G(K)$ orbits on $\hat{F} \times \hat{F} \longleftrightarrow \hat{W}$
$\Sigma_w \longleftarrow w$

$\forall x \in \hat{F}$ Schubert cell $\Sigma_w(x) =$ f.dim affine space $/k$, $\dim = \ell(w)$. — locally compact, so can integrate!

In $\hat{\Xi} \times \hat{\Xi}$ $G(K)$ orbits $\longleftrightarrow \hat{\hat{W}} = \{(w,a) : w \in \hat{W}, a \in L\}$

For $\xi_x \in \hat{\Xi}$ over $x \in \hat{F}$, orbit map
$\hat{\Xi} \supset \Sigma_{w,a}(\xi_x) \xrightarrow{\sim} \Sigma_w(x) \subset \hat{F}$
<u>isomorphism</u>

$\Sigma_{w,a}(\xi)$ "closed" in some sense!



affine Schuberts $\Sigma_w(x)$ not closed

— so theoretically can integrate over the $\Sigma_{w,a}(\xi)$

<u>Lemma</u> $\exists!$ up to scalar measure on $\Sigma_{w,a}(\xi)$ invariant under the stabilizer of ξ ($\simeq \Delta_1$) in $G(K)$.

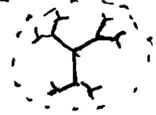
— this explains why we pass to $\hat{\Xi}$.
— i.e. $\Sigma_w(x) \subset \hat{F}$ doesn't have a Stab x-invariant measure

e.g. replace $\hat{F}$ by $\mathbb{P}^1$ 

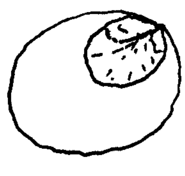
no measure invariant under all of $\mathrm{Stab}\, x$ $\longrightarrow$ must pass to subgroup
$\Longleftrightarrow$ replace x by ξ.

Bruhat-Tits Trees

$G = PGL_2$ $\quad G(k)/G(\mathcal{O}_k)$ = vertices of $\mathcal{T}$ Bruhat-Tits tree.

$q=2$: 

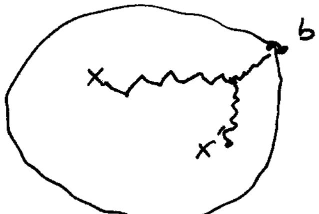
1. Edges out of any vertex form a $\mathbb{P}^1(\mathbb{F}_q)$.
2. Ends of $\mathcal{T} \longleftrightarrow \mathbb{P}^1(k)$
3. $G(k)/D_0 = \{$flags (vertex, edge) in $\mathcal{T}\}$
4. $\Xi = G(k)/D_1 = \{$horocycles in $\mathcal{T}\}$



$$\begin{array}{c} \Xi \\ \downarrow \mathbb{Z} \\ \mathbb{P}^1(k) \end{array}$$

horocycle has a "center" $b \in \partial\mathcal{T} = \mathbb{P}^1(k)$
— points at fixed infinite distance from b.

for $x \in \mathrm{Vert}(\mathcal{T})$, $b \in \partial\mathcal{T}$ $\quad \mathrm{dist}(x,b)$ not defined
but $\delta(x,x',b) =$ "$\mathrm{dist}(x,b) - \mathrm{dist}(x',b)$" well-defined:



two paths to b eventually coincide, say at $y \Rightarrow \delta(y,x',b) = \mathrm{dist}(x,y) - \mathrm{dist}(x',y)$
— independent of choice of $y \in [x,b) \cap [x',b)$

Horocycles = equiv class wrt $x \equiv x'$ for $\delta(x,x',b)=0$
$\Rightarrow$ $\mathbb{Z}$-torsor of horocycles with center b, "Dist_b" $\rightsquigarrow$ "radius" of horocycle.

$H(G(k), \mathcal{O}(k))$ spanned by $T_m : \mathrm{Fun}(\mathrm{Vert}\,\mathcal{T}) \circlearrowleft$

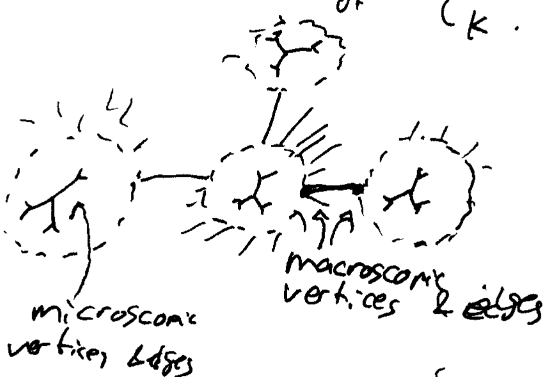
$$(T_m f)(x) = \sum_{\mathrm{dist}(x,y)=m} f(y)$$

- Can form B-T tree over any local field & same properties will hold.

e.g. $(K,k) \rightsquigarrow$ "continuous tree" $\mathcal{T}_K$, with vertices the uncountable set $PGL_2(K)/PGL_2(\mathcal{O}_K)$
Edges out of vertex v $= \mathbb{P}^1_v(k)$ proj line$/k$:
compact profinite space.

Ends of $\mathcal{T}_K = \mathbb{P}^1(K)$

<u>Parshin recommends</u>: realize each $\mathbb{P}^1_v(k)$ as $\partial \mathcal{T}_v$, BT tree for Aut $\mathbb{P}^1_v$, & glue them in to picture of $\mathcal{T}_K$.



$\Rightarrow$ "double tree" $\mathcal{P}$, with "microscopic trees" $\mathcal{T}_v$ ("fibres")

Interesting for $k=\mathbb{R}$: glue in microscopic hyperbolic planes.

Properties: $G(K)$ acts on $\mathcal{P}$

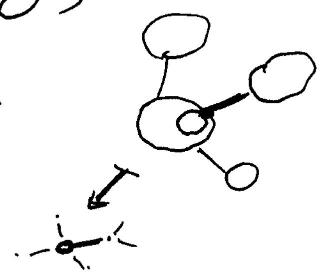
$\{\text{microscopic vertices}\} = G(K)/G(\mathcal{O}')$

$\{\text{microscopic flags}\} = G(K)/\Delta_0$

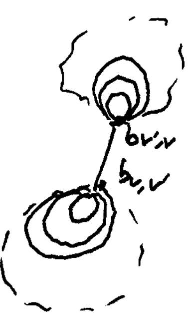
$\hat{\Xi} = G(K)/\Delta_1 = \{\text{microscopic horocycles}\}$

$\downarrow \mathbb{Z}$ — centered at $b \Rightarrow$ get macroscopic vertex + edge = flag.

$\hat{F} = \{\text{flags in } \mathcal{T}_K\}$



$G(K)/\Delta_2 =$ "macroscopic horocycles" - defined by giving distance on $\mathcal{P}$.



<u>Important</u>: If $v, v' \in \mathrm{Vert}(\mathcal{T}_K)$ adjacent $\Rightarrow$ for $b_{v,v'} \in \partial \mathcal{T}_v$ & $b_{v',v} \in \partial \mathcal{T}_{v'}$ we can naturally anti-identify the $\mathbb{Z}$-torsors $\mathrm{Dist}_{b_{v,v'}} \simeq (\mathrm{Dist}_{b_{v',v}})^{\mathrm{dual}}$ $G(K)$-equivariantly

- so if shrink horocycle around $b_{v,v'}$, corresponding horocycle at $b_{v',v}$ grows.

$\Rightarrow$ makes sense to say $\mathrm{dist}(\underset{\mathcal{T}_v}{x}, \underset{\mathcal{T}_{v'}}{x'})$ is of form $\infty + n$, $n \in \mathbb{Z}$

$\Rightarrow$ $G(K)$ invariant distance function on $\mathrm{Vert}_{\mathrm{micro}}(\mathcal{P})$ with values of the form $m\infty + n$ $\quad m \geq 0 \quad n \in \mathbb{Z}$ & if $m = 0$, $n \geq 0$ [i.e. in "positive part" of $\mathbb{Z} \otimes \mathcal{E}$]

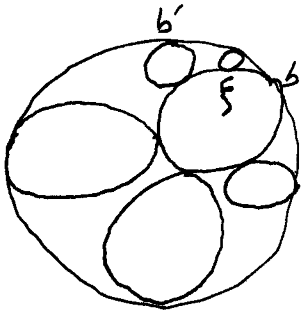
The resulting correspondence between microhorocycles $\mathrm{Dist}_{b_{v,v'}} \simeq (\mathrm{Dist}_{b_{v',v}})^*$ is a Schubert correspondence.

$\xi \in \hat{\Xi} \rightsquigarrow$ micro horocycle centered at b, lies in some $\mathcal{T}_v$

Schubert correspondences $\Sigma_{w,a}(\xi)$:

$\xi + a = \Sigma_{1,a}(\xi)$ 0-dimensional correspondence for $a \in \mathbb{Z}$ $(\ell(a)=0)$

$s \in \mathbb{Z}/2 = W \subset \hat{W} \implies \Sigma_{s,0}(\xi)$ is 1-dimensional - consists of all horocycles with other centers $b' \neq b$ which are <u>tangent</u> to ξ



$= \{ \eta \subset \mathcal{L}_v$ horocycle, center b' : $\xi \cap \eta =$ unique point, on path $b \rightsquigarrow b' \}$

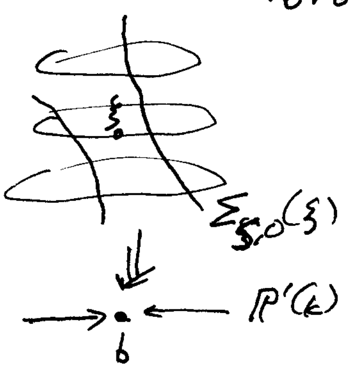
This lifts the Schubert correspondence on $\mathbb{P}^1_v$ which is $\Sigma_s(b) = \{b' : b' \neq b\}$

But as b' approaches b, the radius of the tangent horocycle gets smaller & smaller

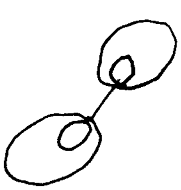
Torsor of horocycles $k^2 \setminus 0 / \mathcal{O}_k^\times$

$$\downarrow$$

$$(k^2 \setminus 0)/k^\times = \mathbb{P}^1(k)$$



- the $\Sigma_{s,a}(\xi)$ are <u>closed</u>, unlike their images in $\mathbb{P}^1$: horocycles are affine lines in $k^2 \setminus 0$, which are closed, but project to non closed subsets of $\mathbb{P}^1(k)$!



$\gamma \in \hat{W}$ non-trivial element of length 0 $\implies$ 0 dim $\Sigma_{\gamma,0}(\xi)$
- corresponds to passing from ξ at b to η of reciprocal radius in neighboring b'.

For any G $\quad \hat{\Xi} = G(k)/\Delta_1 \quad\quad \Sigma_{w,a}(\xi) \simeq \mathbb{A}_k^{\ell(w)}$

$$\downarrow$$

$$\hat{F}$$

$\exists$ measure $\mu_{w,a,\xi}$ on this affine space, invariant under Stab ξ.

Want operators $T_{w,a}$ on some fn. space on $\hat{\Xi}$, given by

$$(T_{w,a} f)(\xi) = \int_{\eta \in \Sigma_{w,a}(\xi)} f(\eta)\, d\mu_{w,a,\xi}$$

---> locally compact but not compact...

$\therefore$ Want our $f : \widehat{\Xi} \to \mathbb{C}$ at least to satisfy

1. $f|_{\forall \Sigma_{w,a}(S)}$ is locally constant
2. ~~f|~~: has compact support.

$\Rightarrow$ $T_{w,a}(f)$ makes sense as a function, and ought to satisfy 1. again, but not nec 2.

$\rightsquigarrow$ trouble composing the $T_{w,a}$'s!

We'll focus on this issue of failure of compact support, which arises even in finite dimensions!

Let's try to define $H(G(k), D_1)$, acting on $\mathcal{F}_0(\Xi)$ = loc constant fns with compact support.

$\Xi = G(k)/D_1$ = loc compact = horocycles in $\mathcal{T}$

$\downarrow$ L-torsor

$F = G(k)/B(k)$ compact profinite

$T_{w,a}$ $w \in W$ $a \in L$ defined in $\mathcal{F}_0(\Xi)$ but take values in $\mathcal{F}(\Xi)$, all loc constant functions.

<u>Principal series intertwiners</u>

$\lambda : L \to \mathbb{C}^*$

$\Rightarrow V_\lambda = \{ f : \Xi \to \mathbb{C}$ loc constant, $f(\xi + a) = \lambda(a) f(\xi)$ $\forall a \in L \}$

$= \Gamma_{cont}(F, \mathcal{L}_\lambda)$ $\mathbb{C}$-line bundle : unramified principal series.

As $\lambda \in Hom(L, \mathbb{C}^*) = T^\vee$ varies, the V_λ form an infinite-dim alg. vector bundle V over $T^\vee$ & the $\Gamma_{reg}(T^\vee, V) = \mathcal{F}_0(\Xi)$ — "spectral decomposition" (~~[illegible]~~)

Easier instance: M an L-torsor. $\lambda \mapsto$ 1-dim vector space $W_\lambda = \{ f : M \to \mathbb{C}$ $f(a+m) = \lambda(a) f(m)$ $\forall a \in L \}$

- form line bundle W on $T^\vee$ (projective module over functions on $T^\vee = \mathbb{C}[L] \dots$)

<u>Claim</u>: $\Gamma_{reg}(T^\vee, W) = \mathcal{F}_0(M)$

Laurent polynomials as functions on tori

coefficients of Laurent polynomials have finite (= compact) support

e.g. $L = \mathbb{Z}$ $M = \mathbb{Z}$ W = trivial line bundle on $T^\vee = \mathbb{C}^*$
$\Gamma(\mathbb{C}^*, W)$ = Laurent polys $\sum_{m \in \mathbb{Z}} a_m t^m \Rightarrow a_- : \mathbb{Z} \to \mathbb{C}$ has finite support

$w \in W$ $b \in F$
$\Rightarrow$ $\Sigma_w(b)$ Schubert cell, $\mathrm{Stab}(b)$ = Borel,
$N(b)$ = radical (stab b) = [Stab b, Stab b]
$\mu_{w(b)} = \{ N_b\text{-invariant measures on } \Sigma_w(b) \}$ is 1-dimensional

As b varies get a $\mathbb{C}$-line bundle on F, μ_w.
<u>Well-known</u> : $\mu_w = \mathcal{L}_{q^{\delta_w}}$ where $\delta_w = \sum_{\substack{\alpha > 0 \\ w\alpha < 0}} \alpha$

$\Rightarrow$ twisted W-action on $T^\vee = \mathrm{Hom}(L, \mathbb{C}^*)$
$w * \lambda = q^\rho w(q^{-\rho} \lambda) = w(q^{-\delta_w} \cdot \lambda)$

So get $M_w(\lambda) : V_\lambda = \Gamma_{rat}(F, \mathcal{L}_{\lambda q^{-\delta_w}} \otimes \mu_w) \longrightarrow V_{w * \lambda}$
$$M_w(\lambda)(f \otimes m)(\xi) = \int_{\eta \in \Sigma_{w,0}(\xi)} f(\eta)\, dm$$

Properties : 1. Converges for λ with $|\lambda^{\alpha^\vee}| > q^{-1}$ $\forall \alpha > 0$ s.t. $w(\alpha) < 0$
and gives a $G(k)$-equivariant operator
2. Depends on λ <u>rationally</u>, so can be defined by analytic continuation even when integral diverges
3. Set $A_w(\lambda) = M_w(\lambda) \cdot \prod_{\substack{\alpha > 0 \\ w\alpha < 0}} \frac{1 - q\lambda^{\alpha^\vee}}{1 - \lambda^{\alpha^\vee}}$

$\Rightarrow A_{ww'} = A_w A_{w'}$
& only singularities of A_w are simple poles on $T_{\alpha,1} = \{\lambda / \lambda^{\alpha^\vee} = 1\}$
for $\alpha > 0$ $w\alpha < 0$

Generic principal series representation:
$V_{rat} = \Gamma_{rat}(T^\vee, V)$ --- a $\mathbb{C}(T^\vee)$ vector space
$\cup$
$\mathcal{F}_0(\Xi) = \Gamma_{res}$
– given a rational function, can expand as power series in many ways & get functions with unbounded support on Ξ

Twisted group algebra for W action (shifted) on $T^\vee$

$\mathbb{C}(T^\vee)[W] = \{ \sum_{w \in W} f_w(\lambda)[w] \}$ $f_w(\lambda)$ rational fns

acts on V_{rat} $\mathbb{C}$-linearly.

__Theorem__ The subalgebra of $\mathbb{C}(T^\vee)[W]$ preserving $\mathcal{F}_0(\Xi) \subset V_{rat}$ is isomorphic to $\ddot{H}_q$

__Proof__ : Residue construction of $\ddot{H}_q$ (Ginzburg-Kapranov-Vasserot)

-- in unshifted language,

$\sum f_w(\lambda)[w]$ s.t. 1. Only singularities of f_w are first order poles on $T^\vee_{\alpha,1}$ $\alpha \in R_+$

& $Res_{T^\vee_{\alpha,1}} f_w + Res_{T^\vee_{\alpha,1}} f_{s_\alpha w} = 0$

2. Each f_w vanishes along $T^\vee_{\alpha,q^2} = \{\lambda / \lambda^{\alpha^\vee} = q^2$ $\alpha > 0$ $w\alpha < 0\}$

Condition 1 ensures that the operators act in $\mathbb{C}[T^\vee]$

-- e.g. BGG / Demazure-Lusztig operators

$$f(x,y) \longmapsto \frac{f(x,y) - f(y,x)}{x-y} = \frac{1}{x-y}[1] - \frac{1}{x-y}[s]$$

- rational coeffs but preserves regular fns.

Condition 2 is extra, needed to guarantee we act on regular <u>sections</u>, not just regular functions

M. Kapranov - Hecke algebras for 2D local fields II 3/12/01

Last time: study $H(G(k), D_1 = T(\mathcal{O}_k)N(k))$ - not compact subgroup, but use theory of principal series intertwiners M_w, $w \in W$

$\Xi_1 = G(k)/D_1$ horocycles in building

$\downarrow$ L-torsor

$F = G(k)/B(k)$ $\qquad \mathcal{F}_0(\Xi)$ loc const fns with cpt support

- module over $\mathbb{C}[L] = \mathbb{C}[T^\vee]$

$M_w : \mathcal{F}_0(\Xi) \longrightarrow \mathcal{F}^w_{rat}(\Xi)$ = functions with support on each fiber lying in a translate of the w-dominant Weyl chamber & whose formal generating series is rational in

$\vdots \; \Xi_b$

$\sum_{a \in L} f(\xi_0 + a) \lambda^a \in \mathbb{C}(T^\vee)$.

$\cdot \; b \in G/B$

But $\mathcal{F}^w_{rat} = \mathcal{F}_0(\Xi) \otimes_{\mathbb{C}[T^\vee]} \mathbb{C}(T^\vee) =: \mathcal{F}^{rat}(\Xi)$

So $M_w : \mathcal{F}_{rat} \longrightarrow \mathcal{F}_{rat}$.

<u>Theorem</u> $\{ \sum_{\substack{\cap \\ \mathbb{C}(T^\vee)}} f_w(\lambda) M_w$ preserving $\mathcal{F}_0(\Xi) \} \simeq \dot{H}_q$

--- Uses residue description of $\dot{H}_q$:

subalgebra in $\mathbb{C}(T^\vee)[W]$ twisted group ring, $\sum f_w(\lambda)[w]$ satisfying:

- Singularities on $T_{\alpha,1} = \{\lambda : \lambda^{\alpha^\vee} = 1\}$ order ≤ 1
- $Res_{T^\vee_{\alpha,1}} f_w + Res_{T^\vee_{\alpha,1}} f_{s_\alpha w} = 0$
- $f_w(\lambda) = 0$ on $T^\vee_{\alpha, q^2}$ for $\alpha \in Dis(w)$ disorder of w ($\alpha \in R_+$, $w(\alpha) \in R_-$)

- match up precisely with properties of M_w to give nonsingular combinations.

<u>Informally</u> Elements of $\dot{H}_q$ <---> some infinite linear combinations of operators T_c corresponding to cosets $c \in \mathcal{D}_1 \backslash G(k) / \mathcal{D}_1 \simeq \hat{W} = W \ltimes L$.

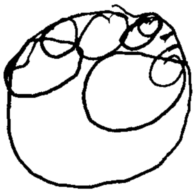
$M_w = T_{(w,0)}$. $T_{(1,a)}$: shift by $a \in L$ on Ξ.

- So finite linear combos of $T_{(w,a)}$ are operators of form $\sum_{w \in W} f_w(\lambda) A_w$ with $f_w(\lambda)$ polynomials.

-- unfortunately this algebra doesn't close, takes $\mathcal{F}_0 \rightsquigarrow$ bigger space...

So we need ∞ combos coming from <u>rational</u> functions (can be expanded in different ways though...)

- integro difference operators : M_w are integral operators, $\tau_{(1,a)}$ shifts are difference operators, & we take special allowable ∞ order combinations of these ....
"pseudo difference" operators...
$\rightsquigarrow$ similar algebra for $G(\mathbb{R})$ gives integro difference operators, shifts come from difference operators on Cartan



Function on space of horocycles with compact support
- integrate over all horo. tangent to it, then add combinations of shorter ones...

Similar description of Cherednik's algebra $\mathcal{H}_q$

Easier: the subalgebra $\ddot{H}_q$ which we will use.

$\check{T}^{aff} = T^\vee \times \mathbb{G}_m$ dual affine torus. $\hat{W} = W \ltimes L$ acts
$\shortparallel$ $\mathrm{Spec}\,\mathbb{C}[L \oplus \mathbb{Z}]$

Def $\ddot{H}_q \subset \mathbb{C}(\check{T}^{aff})[\hat{W}]$ consists of finite combinations $\sum_{w \in \hat{W}} f_w(\lambda)[w]$ with conditions, affine versions of the residue conditions for $\dot{H}_q$ (each f_w has only fin. many poles).

$\ddot{H}_q$ contains almost two copies of $\mathbb{C}[L]$ (from torus & from $\hat{W}$ ...) - one as subalgebra & one as vector space

$\mathcal{H}_q \supset \ddot{H}_q$: enlarge using similar conditions,
- enlarge $\hat{W}$ to $\hat{W}^{ad}$ (for G^{ad} adjoint)

$\hat{W}^{ad} = W \ltimes Q^\vee$, $Q \subset L^\vee$ root lattice, $Q^\vee \supset L$.
[So $\mathcal{H}_q$ contains $\ddot{H}_q$ for G & for $G^\vee$]

& enlarge $\check{T}^{aff}$ to $\tilde{T}^{aff} = \mathrm{Spec}(L \oplus \frac{1}{m}\mathbb{Z})$ m is s.t. $(Q^\vee, L^\vee) \in \frac{1}{m}\mathbb{Z}$
& impose similar residue conditions
$\Longrightarrow$ Cherednik's $\mathcal{H}_q$.

Return to original situation ($K = k((t))$, k, $\mathcal{H}_q$)

$G(K)$ acts on
micro horocycles $\underline{\Xi} = G(K)/\Delta_1 \xrightarrow{P} \hat{F} = G(K)/I_K$ affine flags

$\hat{F}$ inductive limit of compact spaces, but not locally compact

Would like to proceed as before with $\mathcal{F}_0(\hat{\Xi})$ = "functions on $\hat{\Xi}$ with support proper wrt ρ".

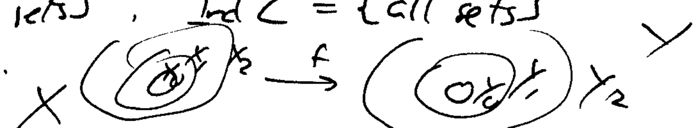
Principal series V_λ, $\lambda \in \hat{T}^{aff}$ etc - just copy previous setting, once we clarify foundational issues...

<u>Foundational issues</u> - What structure do we have on K, $G(K)$?
- What is the exact meaning of $\mathcal{F}_0(\hat{\Xi})$, V_λ etc & what kind of structure do they have?

Ind- & pro- objects!

C category $\Rightarrow$ $Ind(C)$ ind objects: objects are formal symbols "$\varinjlim_{i \in I}$" X_i $(X_i)_{i\in I}$ filtered inductive system over C
& $Hom("\varinjlim_{i\in I}" X_i, "\varinjlim_{j \in J}" Y_j) = \varprojlim_i \varinjlim_j Hom_C(X_i, Y_j)$

<u>Example</u>: C = {finite sets}, $Ind\, C$ = {all sets}
$X = \bigcup_{i \in I} X_i$ finite.



$f: X \to Y$ is compatible system of $f|_{X_i}: X_i \to Y$ ($\varprojlim_I$)
& each $f|_{X_i}$ takes value in some Y_j $\Rightarrow$ $\varinjlim_j$

Similarly f.d. vector spaces $\to$ all vector spaces.
- these are only "interesting" categories in subject, everything else constructed by limits...

$Ind(A\text{-mod}^{f.g.})$ = A-mod
$Ind(Coh_S)$ = $QCoh_S$ quasicoherent

S noetherian quasiprojective separated

Can also realize $Ind\, C \subset Fun^0(C, Sets)$
full subcategory of category of functors
"$\varinjlim$" $X_i \rightsquigarrow \varinjlim h_{X_i}$ limit of representable functors.

$Pro(C) = Ind(C^{op})^{op}$
- objects are formal inverse limits "$\varprojlim_{i\in I}$" X_i

$Hom("\varprojlim_{i \in I}" X_i, "\varprojlim_{j \in J}" Y_j) = \varprojlim_J \varinjlim_I Hom_C(X_i, Y_j)$

Example C = {finite sets} $Pro(C)$ = compact totally disconnected spaces & continuous maps, eg $\mathbb{Z}_p$, $\mathbb{P}^n(\mathbb{Q}_p)$

Here $\varprojlim_j \varinjlim_i$ corresponds to $\forall \varepsilon\ \exists \delta$ in def of continuity!
$\forall j\ \exists i$ & map $X_i \to Y_j$

Now can form Ind(Pro C) etc..
- e.g. $\mathbb{Q}_p = \varinjlim_n \varprojlim_m p^{-n}\mathbb{Z}_p/p^m\mathbb{Z}_p$.

Locally compact ind-pro objects:
- want to distinguish $\mathbb{Q}_p$ from affine flags $\mathcal{F}(\mathbb{Q}_p) = \varinjlim_j$(Schubert varieties over $\mathbb{Q}_p$)
— both ind-profinite but latter is not locally compact.

- want to include among locally compacts objects of form
"$\varinjlim_i$" "$\varprojlim_j$" X_{ij} where (X_{ij}) is an ind-pro system
... have diagrams $i \le i'$, $j \le j'$

$$\begin{array}{ccc} X_{ij'} & \longrightarrow & X_{i'j'} \\ \downarrow & & \downarrow \\ X_{ij} & \longrightarrow & X_{i'j} \end{array}$$

commutative & want all such diagrams Cartesian

- locally compacts ought to be objects of both ~~Ind~~ Pro C & Pro Ind C ...

Sample statement: locally compact totally disconnected spaces $\longleftrightarrow$ loc compact ind-pro objects in finite sets.
(ind direction should be strict: ind maps monomorphisms)

$\exists$ good theory for linear case, when C is an exact category $\longrightarrow$ new exact category $\mathcal{L}(C)$ (Kato, Beilinson "$\varinjlim\varprojlim C$")

Ex. $\mathbb{Q}_p \in \mathcal{L}$(finite ab. groups)
$K = k((t)) \in \mathcal{L}$(f.d. k-vect spaces) $\subset \mathcal{L}(\mathcal{L}$(fin ab grps))

Important: K is ring object in the ind-pro category..
- not tensor category though (get iterated Laurents if try to tensor)

NB K with the topology of $\varinjlim_n \varprojlim_m t^{-n}\underbrace{k[[t]]/t^m k[[t]]}_{\text{f.d. p-adic variety topology}}$
is not a topological ring!
- mult. not continuous... so can't topologize $G(K)$.

But $G(K)$ is a group object in $(\text{Ind-Pro})^2$ (finite sets)

Ind-Pro vs Pro-Ind: $\mathcal{L}(C)$ can also be realized (full subcategory) inside $\text{Pro}(\text{Ind } C)$, $\text{"}\varprojlim_j\text{" "}\varinjlim_i\text{"} X_{ij}$

- Can't generally compare ProInd & IndPro, but each contains full subcategory $\simeq \mathcal{L}(C)$.

<u>Exercise</u> Let (X_{ij}) be any ind-pro diagram of sets.

- Construct a natural map $\varprojlim_j \varinjlim_i X_{ij} \xleftarrow{\varphi} \varinjlim_i \varprojlim_j X_{ij}$
- Prove that if squares are Cartesian then φ is a bijection

<u>Probably theorem</u>: Locally compact totally disconnected spaces $= (\text{Ind Profinite}) \cap (\text{Pro Ind finite}) \subset$ top. spaces.
(or Ind compact) (= prodiscrete)

- Pro ind not actually embedded into Top: can have <u>empty</u> proj limit of surjective system.

<u>Examples</u>: $\hat{F}$ is ind-pro object in finite sets

If $\Gamma \subset G(K)$ congruence subgroup $\Rightarrow$

$G(K)/\Gamma \longrightarrow (G(K)/I)_{\hat{F}}$ has <u>locally</u> compact fibers
$= \varinjlim \Sigma_w$ schubert varieties

So $G(K)/\Gamma = \text{"}\varinjlim\text{"}$ (loc. compact spaces $\subset$ Ind-Pro-finite)

$\Rightarrow G(K) = \text{"}\varprojlim_{\Gamma}\text{"}\, G(K)/\Gamma$ is a group pro-ind object in locally compact spaces.

$G(K)$ is loc compact object of ind-pro (loc compact)
- same for any alg variety, taken over K.

What is analog of <u>smooth</u> representations?

p-adic case: $\pi : G(k) \longrightarrow \text{End}(V)$

$G(k) = \varprojlim_{\Gamma \text{ cpt open}} G(k)/\Gamma$ (discrete ∞ set)

$\text{End}(V) = \text{"}\varprojlim_i\text{" } \text{"}\varinjlim_j\text{" } \text{Hom}(V_i, V_j)$
∞-dim vector spaces

$V = \varinjlim V_i \in \text{Ind}(\text{Vect}_{fd})$

π is <u>smooth</u> iff it's a morphism of pro-objects in sets ($\Leftrightarrow$ pro-ind objects in finite sets)

$$\text{"}\varprojlim\text{"}\, G(k)/\Gamma \longrightarrow \varprojlim_{\substack{V' \subseteq V \\ \text{f.d.}}} \mathrm{Hom}(V', V) \quad : \text{ i.e. } \forall V' \exists \Gamma \text{ s.t. } \pi(\gamma)\ \gamma \in \Gamma \text{ fix } V'$$

so $G(k) \longrightarrow \mathrm{Hom}(V', V)$ factors through $G(k)/\Gamma$.

For 2dim local case $G(K)$ double proind in finite sets

$\longrightarrow$ look for actions on pro-ind (or ind pro) objects V in Vect_{fd} ! For such V, $\mathrm{End}\, V$ is double proind object & $\pi : G(K) \longrightarrow \mathrm{End}\, V$ is smooth if π is a morphism of such objects.

Example Unramified principal series of $G(K)$

$\hat{\Xi}$ $\hat{\Xi}_\alpha$

$\downarrow L$ $\downarrow L$

$\hat{F} = \varinjlim F_\alpha$ Schubert

$V_\lambda = \text{"}\varprojlim\text{"}\, \{\lambda\text{-homog. locally constant fns on } \Xi_\alpha\} \in \mathrm{Pro\,Ind\,Vect}_{fd}$

$$\mathcal{F}_0(\hat{\Xi}) = \text{"}\varprojlim\text{"}\, \mathcal{F}_0(\Xi_\alpha) \in \mathrm{Pro}(\mathbb{C}[T^\vee]\text{-mod}_{fg})$$

Claim V_λ is smooth in the above sense.

First step: $\forall$ Schubert variety in $\hat{F}$ stabilized by some congruence subgroup in $G(\mathcal{O}_K)$

Now for $w \in \hat{W}$ define $M_w : V_\lambda \longrightarrow V_{w\lambda}$ by integrals over f.dim Schubert cells (= the Hecke operators for Δ_1)

- morphism of pro-vector spaces (when convergent) rational dependence on λ.

$\rightsquigarrow$ H_q acts on $\mathcal{F}_0(\hat{\Xi})$ by morphisms of pro $\mathbb{C}$-vector spaces commuting with $G(K)$ action