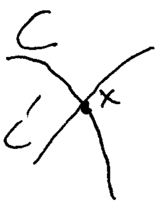
M. Kapranov - Towards a Higher-Dimensional Langlands Program 5/15/00

III. (1) Naive

$X/\mathbb{F}_q$ surface (smooth, proj). $\mathcal{L}$ unramified ${}^{L}G$ local system ($\mathbb{C}=\overline{\mathbb{Q}}_\ell$)

$\forall$ smooth curve $C\subset X \Rightarrow \mathcal{L}|_C$ local system on C.

By usual Langlands $\Rightarrow f_{\mathcal{L},C} : Bun_G(C) \to \mathbb{C}$ automorphic function.



How are the $f_{\mathcal{L},C}$ related for different C? e.g. for $P\in Bun_G(X)$, $f_{\mathcal{L},C}(P|_C)\ldots$

Hecke operators on C come from $Gr(C,x)\neq Gr(C',x)$

When $G=GL_1$, as a reduced set $Gr(C,x)_{red}=\mathbb{Z}$

$\Rightarrow$ define CH_0 by using identifications from curves..

but infinitesimally this is un-natural (not as sets).

Abelian case $G=GL_1$ $\mathcal{L}\xleftrightarrow{\text{2d CFT}}\chi_{\mathcal{L}} : CH_0(X)\to\mathbb{C}^*$ character.

$P\in Pic\,X$, have direct image $i_C : Pic\,C\to CH_0\,X$

$\Rightarrow f_{\mathcal{L},C}=\chi_{\mathcal{L}}\circ i_C$

$i_C(P|_C)=[P]\cdot[C]$ intersection $Pic\,X\otimes Pic\,X\to CH_0 X$

So $f_{\mathcal{L},C}(P|_C)=\chi_{\mathcal{L}}([P]\cdot[C])$ depends only on linear equivalence class of C.

- not true in nonabelian case, at least even for $\mathcal{L}$ reducible & $C=\mathbb{P}^1$ varying in a family: Eisenstein series on $\mathbb{P}^1$ is a function of splitting type, which can vary...

In general: we'd like to "nonabelianize" $CH_0(X)$

Possibilities: a) View CH_0 (surface) as $H^2(\underline{K_2})$ & nonabelianize.

b) View CH_0 as target of characteristic classes of elements from $H^1(\underline{G})$ as nonabelianization.

c) relate Hecke operators on G-bundles on different curves

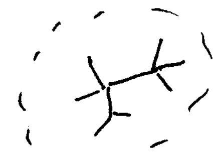
[Parshin: construct double buildings ~ double affine Grassmannian: $\tilde{\mathcal{O}}_E\subset E\supset\mathcal{O}_E$

$k_{E_1}\supset\mathcal{O}_{E_1}\to\mathbb{F}_q$

$G(E)\supset II=\{g\in G(\tilde{\mathcal{O}})\mid R(g)\subset B(\mathbb{F}_q)\}$ double Iwahori

$E_1=\mathbb{Q}_p$. E local field with residue $\mathbb{Q}_p$

$\Rightarrow$ building  star $=\mathbb{P}^1(\mathbb{Q}_p)$ —

now replace each vertex by tree of E_1  $\partial = \mathbb{P}^1(\mathbb{Q}_p)$..

2. <u>Various levels of Hecke operators</u> $G = GL_n$

vector bundles $\subset$ coherent sheaves.

$X/\mathbb{F}_q$ smooth proj $\dim = n$.

$\forall\, \mathcal{A} \in \mathrm{Coh}\, X$ define Hecke operator $T_{\mathcal{A}} : \mathbb{C}[\mathrm{Coh}\, X] \circlearrowleft$

$$(T_{\mathcal{A}} f)(\mathcal{F}) = \sum_{\substack{\mathcal{F}' \subset \mathcal{F} \\ \mathcal{F}/\mathcal{F}' \simeq \mathcal{A}}} f(\mathcal{F}') \quad \text{.. makes sense.}$$

<u>Composition</u>:

$$T_{\mathcal{A}} \circ T_{\mathcal{B}} = \sum_{\mathcal{C}} g_{\mathcal{A}\mathcal{B}}^{\mathcal{C}}\, T_{\mathcal{C}}, \quad g_{\mathcal{A}\mathcal{B}}^{\mathcal{C}} = \#\left\{ \mathcal{A}' \subset \mathcal{C} \,\middle|\, \begin{array}{l} \mathcal{A}' \simeq \mathcal{A} \\ \mathcal{C}/\mathcal{A}' \simeq \mathcal{B} \end{array} \right\}$$

– Hall algebra, basis $[\mathcal{A}]$ & multiplication $[\mathcal{A}][\mathcal{B}] = \sum g_{\mathcal{A}\mathcal{B}}^{\mathcal{C}} [\mathcal{C}]$

$H(\mathrm{Coh}\, X)$.

Frobeniuses $\longleftrightarrow$ points $x \in X$ .. but for $\dim X > 1$
this doesn't preserve vector bundles: $T_{\mathcal{O}_x}$ only add singularities to sheaves -- don't get eigenfunctions...

Better: $\mathcal{A}$ = vector bundle on a hypersurface,
$T_{\mathcal{A}}$ preserves $\mathbb{C}[\mathrm{Bun}(X)]$.

For $m \leq n$ define $\mathrm{Coh}_m(X) = \{ \mathcal{F} \in \mathrm{Coh}\, X \mid \mathrm{Supp}\, \mathcal{F}$ has $\dim \leq m$
& $\forall x \in X \;\; \mathrm{Ext}_x^i(\mathcal{O}_x, \mathcal{F}) = 0 \;\; i < m \}$

(<u>Cohen-Macaulay</u> condition)

· ~~semicontinuity applied to $\mathcal{F}$ sits into~~ Stable under direct image for finite morphism.

<u>Ex</u> (1) $\mathrm{Coh}_{\dim X}(X) = \mathrm{Bun}(X)$

$\mathrm{Coh}_0(X)$ = sheaves with 0-dim support

$\mathrm{Coh}_1(X)$ = " " 1-dim " & no 0-dim torsion

<u>Lemma</u> If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$,
$\mathcal{F}'' \in \mathrm{Coh}_{m-1}$, $\mathcal{F} \in \mathrm{Coh}_m \Rightarrow \mathcal{F}' \in \mathrm{Coh}_m$

<u>Corollary</u> The Hall algebra $H(\mathrm{Coh}_m)$ acts on $\mathbb{C}[\mathrm{Coh}_{m+1}]$ by Hecke operators.

Any $f \in \mathbb{C}[\mathrm{Coh}_m]$ (with good properties..) $\rightsquigarrow$

$$T_f = \sum_{\mathcal{A} \in \mathrm{Coh}_m} f(\mathcal{A}) \cdot T_{\mathcal{A}}$$

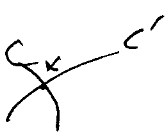
Example $n=2$ X surface : $\mathcal{L}$ rank r unramified local syst.
$C \subset X$ $f_{\mathcal{L},C} : Bun_r(C) \to \mathbb{C}$ eigenfunction of $H(Coh_0)$
$$\Rightarrow \quad T_{f_{\mathcal{L},C}} = \sum_{E \in Bun_r(C)} f_{\mathcal{L},C}(E)\, T_E \quad \text{acts on } \mathbb{C}[Bun_N(X)]$$

Sum only goes in one direction, towards negative bundles, looks ~ geometric sum like L-function... ⇒ makes sense

Q. What are algebraic relations among the $T_{f_{\mathcal{L},C}}$ for different C
(can formally compose..)
-expect eigenvalues relate to L-function of restriction, like eigenvalues in 1-dim related to Frobenius ("ordin L-fn")

Simplest case: $r=1$ (But $T_{f_{\mathcal{L},C}}$ acts on Bun_N, $N \geq 1$!)

$\Rightarrow \chi : CH_0(X) \to \mathbb{C}^*$, $i_C : Pic(C) \to CH_0(X)$
$$T_{f_{\mathcal{L},C}} = \sum_{L \in Pic\, C} \chi(i_C(L))\, T_L$$



Claim In this case the $T_{f_{\mathcal{L},C}}$ commute for curves intersecting transversally.
Pf Let $C \cap C' \ni x$. Want in $H(Coh_1(X))$ & show that $\Phi_C = \sum_{L \in Pic\, C} \chi(i_C(L))\, [L]$ commutes with $\Phi_{C'}$

Possible extensions of L by M :
a) $L \oplus M$, has unique splitting, ∃! subsheaf M with quotient L
$g^{L\oplus M}_{L,M} = g^{L\oplus M}_{ML} = 1$
b. Nontrivial extension $\mathcal{E}(L,M)$ which is line bundle on $C \cup C'$ $\quad 0 \to M \to \mathcal{E}(L,M) \to L \to 0$
--- L is uniquely defined by $\mathcal{E}(L,M)$: $L = \mathcal{E}(L,M) \otimes \mathcal{O}_C$,
& M is kernel of this morphism...
$g^{\mathcal{E}(L,M)}_{ML} = 1$

However : $0 \to L(-x) \to \mathcal{E}(L,M) \to M(x) \to 0$
by restriction to C' instead of C....
$g^{\mathcal{E}(L,M)}_{L(-x)M(x)} = 1 \quad \Rightarrow$
$$\chi(i_C(L(-x))) = \chi(i_C(L))\,\chi(x)^{-1}$$
$$\chi(i_{C'}(M(x))) = \chi(i_{C'}(M)) \cdot \chi(x)$$

So the coefficient at any $[\mathcal{E}]$, $\mathcal{E}\in \mathrm{Pic}^{\cdot}(C\cup C')$ in $\Phi_C\Phi_{C'}$ & $\Phi_{C'}\Phi_C$ is the same $\Longrightarrow$ operators commute ▨

More generally: A torsion-free sheaf $\mathcal{F}$ on $C\cup C'$ is obtained from $E\in \mathrm{Bun}_r(C)$, $E'\in \mathrm{Bun}_{r'}(C')$ & subspaces $W\subset E_x$, $W'\subset E'_x$ & an isomorphism $W\xrightarrow{\sim} W'$. $\Rightarrow$ identify W, W' in $E\oplus E'$.

$0\to \text{kernel} = \beta_{C'}(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow \alpha_C(\mathcal{F}) = \mathcal{F}\otimes\mathcal{O}_C/\text{tor} \to 0$

$0 \longrightarrow \alpha_{C'}(\mathcal{F})$ & $\beta_C(\mathcal{F})$ similarly.

Note that $\beta_C(\mathcal{F})$ is an elementary modification of $\alpha_C(\mathcal{F})$ along a subspace $\Lambda\subset\alpha_C(\mathcal{F})_x$ (image of $\mathcal{F}_x$)

∴ sections of $\beta_C(\mathcal{F})$ = those of $\alpha_C(\mathcal{F})$ lying, over x, in Λ.

Similarly for $\beta_{C'}$, $\alpha_{C'}$.

$\mathrm{Coh}_{r,r',i}(C\cup C')$ = torsion-free sheaves on $C\cup C'$ of gen rank r, r', $i = r+r'-\dim\mathcal{F}_x$

$\mathrm{Coh}_{r,r',i}(C\cup C')$ sits in diagram:

$$\begin{tikzcd}
 & \mathrm{Bun}_r C & & \mathrm{Bun}_{r'} C' & \\
\mathrm{Hecke}_{r-i,x}(C) \arrow[ur] \arrow[dr] & & \mathrm{Coh}_{r,r',i}(C\cup C') \arrow[ul, "\beta_C"'] \arrow[ur, "\beta_{C'}"] \arrow[ll, "\exists"'] \arrow[rr, "\exists"] \arrow[dl, "\alpha_C"] \arrow[dr, "\alpha_{C'}"] & & \mathrm{Hecke}_{r'-i,x}(C') \arrow[ul] \arrow[dl] \\
 & \mathrm{Bun}_r C & & \mathrm{Bun}_r(C') &
\end{tikzcd}$$

3) Attempts at nonabelianizing K_n, H^n

E = n-dim local field

1-dim rep ρ of Weil group $W(E)$ $\Rightarrow$ character

$\chi_\rho : K_n^M(E) \longrightarrow \mathbb{C}^*$

Suslin: $K_n^M(E) = \dfrac{H_n(GL_n F, \mathbb{Z})}{\mathrm{Im}\, H_n(GL_{n-1} F, \mathbb{Z})}$

– so χ_ρ gives a class $c_\rho \in H^n(GL_n(F), \mathbb{C}^*)$

What if $\dim\rho > 1$? e.g. $\rho = \rho_1\oplus\rho_2$

$n=2$, G any group, A abelian group

$H^2(G,A) = \{A\text{-groupoids with } G\text{-action}\}/\text{equiv}$

category $\mathcal{C}$ with $\mathrm{Hom}_{\mathcal{C}}(x,y)$ made into A-torsors, compatibly

G action: $g \in G \Rightarrow \varphi_g : C \to C$

$g_1, g_2 \Rightarrow \psi_{g_1,g_2} : \varphi_{g_1} \circ \varphi_{g_2} \Longrightarrow \varphi_{g_1 g_2}$ natural transf

+ cocycle condition for g_1, g_2, g_3.

$A = \mathbb{C}^*$ A-torsor $\leftrightarrow$ 1-dim $\mathbb{C}$ vector space L

($\mathbb{C}$-Vect, $\otimes$) ringed category

$- \otimes L$ is self-equivalence of $\mathbb{C}$-Vect as module category over itself

$\Rightarrow H^2(G, \mathbb{C}^*) = \{$ "representations" of G in $\mathbb{C}$-vect by module category self-equivalences .. $\}$

Natural idea: Look at ($\mathbb{C}$-Vect)-module categories with $GL_2(E)$-action

"Example": "principal series" for a 2-dim local field E,

$\rho_1, \rho_2 : W(E) \to \mathbb{C}^* \quad \rightsquigarrow \lambda_1, \lambda_2 \in H^2(GL_2(E), \mathbb{C}^*)$

realize as functors $\varphi_1, \varphi_2 : GL_2 E \to \{$self-equiv of $\mathbb{C}$-Vect$\}$

Conceptually have functors

$\Phi_1, \Phi_2 : \{$2-dim vect space $/E\} \longrightarrow$ module category $/\mathbb{C}$-Vect, free of rank 1.

Like $\otimes$ of module categories $\leftrightarrow$ addition of cohomology class.

Consider the Grassmannian $G(2,4)$ over E

$\{V \subset E^4 \mid \dim V = 2\} \rightsquigarrow \exists$ "stack of categories" $\mathcal{S}$ (analog of line bundle) on $G(2,4)(E)$

$V \rightsquigarrow \Phi_1(V) \otimes \Phi_2(E^4/V)$

analog of principal series = sections of line bundle, comes from character of group over local field..

"Principal series reps" = category of $\Gamma_{cont}(G(2,4)(E), \mathcal{S})$ with $GL_4(E)$ action.

Problems: i) not continuous construction: cocycles representing classes λ_i not quite continuous...

ii) Topology on E & $G(2,4)(E)$ not good--- replace by ind-pro structure.

4) ~~Geometric attempts at nonabelian~~ $H^2(X, G)$

X surface, G alg. group.

$X = U_0 \cup U_1 \cup U_2$ need 3 open subsets (e.g. flag covering)

G abelian $\Rightarrow$ $H^2(X, G)$ defined:

$$\frac{G(U_{012})}{G(U_{01})\,G(U_{02})\,G(U_{12})}$$

write $G_{ij} = G(U_{ij})$, $G_i = G(U_i)$

G nonabelian: can't factorize by three subgroups -- only have two sides

(groups $\leftrightarrow$ matrices M_i^j tensors i — M — j, j — N — k

$\Rightarrow$ — M — N — .. could replace by M

& act by —o— 's on such..

Tensors $(\mathbb{C}^2)^{\otimes 3}$ is prehomogeneous vector space for $GL_2 \times GL_2 \times GL_2$ — inside have open orbit Γ ... $\Rightarrow$

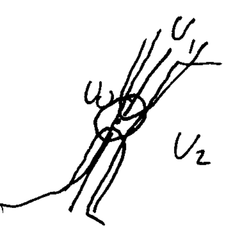
can take $GL_2(U_{02}) \backslash \Gamma(U_{012}) \backslash GL_2(U_{01}) / GL_2(U_{12})$ — three actions commute.)

We can form $G_{01} \backslash G_{012} / G_{02} = H^1(U_0 \cap (U_1 \cup U_2), G)$

For G abelian, this maps to $H^2(X, G)$, but makes sense for all G.

analytic setting: this is $Bun_G(U_{01} \cup U_{02})$ (Stein setting.)

~~in formal setting~~



Let $\{U_i\}$ = Parshin covering corresponding to a flag $x \in C \subset X$

$U_{01} \cup U_{02}$ = punctured formal (or small) disc at x $\mathcal{D}_x^{\circ} = \mathcal{D}_x - \{x\}$

- so we're looking at singularities of bundles...

$$G_{01} \times G_{02} \times G_{12} \xrightarrow{\delta} G_{012} \quad \text{coboundary map (for } Bun_G = \text{nonab. } H^1)$$

$$(\varphi_{01}\ \varphi_{02}\ \varphi_{12}) \longmapsto \varphi_{02}^{-1}\varphi_{12}\varphi_{01}$$

$G_0 \times G_1 \times G_2 \circlearrowleft$ acts: $\varphi_{01}' = \varphi_1 \varphi_{01} \varphi_0^{-1}$ etc,

preserving cocycles $\delta^{-1}(1)$.

$\delta^{-1}(1) / \prod G_i = Bun_G(X)$.

Take $\lambda \in G_{012}$ & look at $\delta^{-1}(\lambda)$: cochains here represent "fake" bundles - but not preserved by $G_0 \times G_1 \times G_2$ - unless G is abelian

($\Rightarrow$ elements of Brauer group as representing fake bundles)

NB: $\delta^{-1}(\lambda)$ always preserved under $G_1\times G_2$:

$\varphi_{12}\varphi_{01} = \varphi_{02}\cdot\lambda$

$\rightsquigarrow \psi_2\varphi_{12}\psi_1^{-1}\psi_1\varphi_{01}\psi_0^{-1} \overset{?}{=} \psi_2\varphi_{02}\psi_0^{-1}\lambda$

- OK if $\psi_0 = 1$.

$\delta^{-1}(1)/G_1\times G_2 = Bun_G(X, U_0)$ bundles trivialized on $U_0 = D_x$.

(can write coboundary in 3 different ways, get 3 different invariances $\Rightarrow Bun_G(X, U_i)$ $i = 1,2,3$)

$= \{P,\tau : P|_{D_x} \to D_x\times G\} = \{P^0 \in Bun_G(X\setminus x),\ \tau : P^0|_{D_x^0} \to D_x^0\times G\}$

$\lambda \longmapsto [\lambda] \in G_{01}\backslash G_{012}/G_{02} \Rightarrow Q_\lambda \in Bun_G(D_x^0)$

$\delta^{-1}(\lambda)/G_1\times G_2 = \{P^0\in Bun_G(X\setminus x),\ \tau : P^0|_{D_x^0} \longrightarrow Q_\lambda\}$
$=: Bun_G(X,x)_\lambda$

$$\begin{tikzcd}
 & \overline{M} = G_1\backslash G_{12}/G_2 = \coprod_{Q\in Bun_G(D_x^0)} Bun_G(X\setminus x; U)_Q \arrow[d, "\overline{\pi}"] \\
\frac{G_{01}\times G_{12}\times G_{02}}{G_1\times G_2} \arrow[ur] \arrow[d, "\pi"] & \\
G_{012} \arrow[r] & Bun_G(D_x^0) = G_{01}\backslash G_{012}/G_{02}
\end{tikzcd}$$

-- family, with $\overline{\pi}^{-1}([1]) = Bun_G(X, U_0)$

-- formal neighborhood of $\overline{\pi}^{-1}(1)$ makes sense for actual (formal) Parshin covering. .. note tangent at 1 is $\mathfrak{g}_{01}\backslash\mathfrak{g}_{012}/\mathfrak{g}_{02}$

Suppose $(P,\tau)\in\overline{\pi}^{-1}(1)$.

Coker $d_{(P,\tau)}\overline{\pi} = H^2(X, \mathrm{ad}\, P)$ - obstruction space to deformations of bundle.

$Bun_G^\circ(X, D_x) = \{(P,\tau) \mid H^2(X, \mathrm{ad}\, P) \neq 0\}$

-- smooth part of this acquires a deformation

$T_{[1]} Bun_G(D_x^0) = H^1(D_x^0, \mathfrak{g}) \xrightarrow[\text{Kodaira Spencer}]{} H^1(Bun_G^\circ(X, U_0), T)$

[For curves $C \ni x$, $D_x \Rightarrow$

$H^0(D_x^0, \mathfrak{g}) \longrightarrow H^0(Bun_G(X, D_x), T)$

$\cup|$ Kac-Moody

vector fields on moduli ..]

Derived deformation theory (Drinfeld, Kontsevich, ...)

moduli spaces should be replaced by derived versions, which are dg schemes = (schemes, sheaf of dg algebras)

- which should always be smooth in appropriate sense:
tangent spaces $\rightsquigarrow$ complexes,
& KS map quasi-isomorphism in all degrees.

Ciocan-Fontanine-Kapranov: Derived Quot Schemes math AG/99 ...

$\Rightarrow$ replace $\pi^{-1}(1)$ by derived fiber:
$f: X \to Y \ni y$ .... $f^{-1}(y)$ has $\mathcal{O}_{f^{-1}(y)} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{O}_y$

replace by "homotopy fiber" $Rf^{-1}(y) = (\underset{\text{space}}{f^{-1}(y)}, \mathcal{O}_X \overset{L}{\otimes}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{O}_y)$

.. always have K-S map:
$T_y Y \longrightarrow \mathbb{H}^1(Rf^{-1}(y), T^\bullet)$ tangent complex
(really $RT_y Y \to R\Gamma(Rf^{-1}(y), T)[1]$

In our case, consider $R\delta^{-1}(1)/G_0 \times G_1 \times G_2 = RBun_G(X)$ (smooth dg stack)

$\frac{R\delta^{-1}(1)}{G_1 \times G_2} = RBun_G(X, \mathcal{D}_x)$ dg scheme of infinite type.

"Theorem" There is a morphism of dg-Lie algebras
$R\Gamma(\mathcal{D}_x^\circ, \mathfrak{g}) \longrightarrow R\Gamma(RBun_G(X, \mathcal{D}_x), T^\bullet)$
for X of any dimension
$\to$ higher Kac-Moody... $\dim X = n$ have $H^0 = \mathfrak{g}[[t_1 \dots t_n]]$,
$H^{n-1} \sim \mathfrak{g}[t_1^{-1}, \dots t_n^{-1}]$ local cohomology supported on point...