

A. Kapustin - Magnetic Wilson Loops 9/25/05
hep-th/0501015 & Montonen-Olive duality

Abelian gauge theory: F 2-form gauge field strengths
 \downarrow
 $*F$ dual: symmetry of gauge theory

Is there a similar symmetry for YM? Not classical
symmetry of equations of motion.

Goddard-Nuyts-Olive 1977: study gauge fields
with R decay & spherical symmetry (~~symmetry~~)
- classified by weights for G dual group
eg $Sp \leftrightarrow SO(2n+1)$

Montonen-Olive '77: Consistency of G & G^* gauge theory
taking coupling $e^2 \leftrightarrow \frac{1}{e^2}$.

Witten-Olive, Osborne: this can work only in SUSY theories.

$N=4$ SYM: spectrum of monopoles has right quantum numbers to be symmetric $G \leftrightarrow G^*$
for lump-like solutions ... have to be careful with topology of gauge field at infinity.

θ -angle shift $\Theta \sim \Theta + 2\pi$ doesn't come up with $e^2 \leftrightarrow \frac{1}{e^2}$
 \Rightarrow set $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$ complexified gauge coupling.

Symmetry is $\begin{cases} \tau \leftrightarrow -\frac{1}{\tau} & \text{Montonen-Olive} \\ \tau \leftrightarrow \tau + 1 & \text{SUSY} \\ \rightarrow SL_2 \mathbb{Z} \text{ symmetry} \end{cases}$

Would like to understand resulting maps on operators
in the dual theories! In particular Wilson loops

E.g. F_4, G_2 self-dual Lie algebras: find $\tau \leftrightarrow \frac{1}{\tau}$ map
correct in F_4 : $\tau \rightarrow \frac{1}{2\tau}, G_2 \tau \rightarrow \frac{-1}{2\tau}$

-- find subgroups of $SL_2 \mathbb{R}$ generated by L
are not subgroups of $SL_2 \mathbb{Z}$... 'Hecke groups'

Wilson loops : $W_R(C) = \text{tr}_R \text{ Prod}[S_A]$

R representation, C curve ... some as order parameters
for confinement etc.

What is dual of Wilson loops?

Quasi answer: 't Hooft defined magnetic charge
of Wilson loops, but not full solution of the
duality

't Hooft: given loop in \mathbb{R}^4 look at path integral
over gauge fields on \mathbb{R}^4 - loops with fixed
behavior around the loop : transversal topology
of S^2 , classify topology by $\pi_1(G)$

---> get elements labelled by π_1 of G
eg $SU(n)/\mathbb{Z}_n \rightarrow$ n distinct answers

--- rough classification of operators, only topological

We'll describe duals of Wilson loops as certain
singularities of gauge fields.

$N=2$ SYM $G = SU(2)$, four flavors of hypermultiplets
 $N_f = 4$ ($\in 2$ of $SU(2)$)

$N=4$ all fields in adjoint rep don't feel SU_n/\mathbb{Z}_n
but in $N=2$ have really SU_n not SU_n/\mathbb{Z}_n
 \rightarrow no π_1 , don't see 't Hooft loops
So what happened to Wilson loops under duality??

Seiberg: $N=1 \quad G = SU(N_c)$,
 N_f fields in N_c of $SU(N_c)$
 $\hookrightarrow N_f \in \bar{N}_c$ of $SU(N_c)$
 \Downarrow and
 $N=1 \quad G = SU(N_f - N_c) \dots$

What happens to Wilson loops under duality?
 Went map to Wilson loops, expect also
 see electric/magnetic duality, but have no Π , ...

Operators in QFT: e.g. 2d Free Fermion Theory
 here twist operators σ, μ : not really
 fractions of ψ : keep action but integrate
 over it's with singularity at a given point (insertion)
 --- need not just topology but asymptotics
 of gauge field. Still local operator: concentrated at one

3d gauge theory! Insert a source at a point!
 look at fields with magnetic flux
 around this point. Need to specify more than topology

R⁴

e.g. in 4d if excise a point no non-trivial topology
 of gauge fields on $S^3 \dots$ but in gauge theory
 have no point operators

In 4d excise curve, has transversal S^2 boundary!

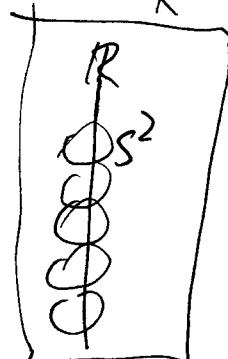
Wilson loops: line $e^{\int A dt}$

symmetries of metric

(conformal rescaling of metric
 along curve)

preserves $\frac{dt^2 + dr^2}{r^2} + d\Omega^2 = AdS^2 \times S^3$

$$SL(2, R) \times Spn 3$$



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2d version \Rightarrow metric rescaled $\Rightarrow R \times S^2$ Plot
 radial quantizn: operator becomes a state
easier to see symmetry.

Abelian version (Maxwell theory) : What are natural boundary conditions to impose $r=0$ on $AdS^2 \times S^2$? can have unrestricted boundary cond. bcs
 $F = dA \sim a(t) \frac{dr}{r^2} + O(r)$

$A \sim a(t) \frac{dt}{r}$ gauge field arbitrary at boundary
 (free boundary)

(or also insert weight in path integral $e^{\int_{\text{free}} A_0 dt}$)

$$A_{cl} = a_{cl} \frac{dt}{r} \quad a_{cl} = \frac{i e^2}{4 \pi} n$$

Magnetic version : $m = G$ of gauge field $\sim S^2$

$$F = \frac{m}{2} \text{vol}(S^2) \tau - \dots \text{ near } 2A \cdot S^2$$

$$F_{ij} = \frac{m}{2} \frac{\epsilon_{ijk} \times k}{r^3} - \dots$$

mixed electro-magnetic : $F = \frac{m}{2} \text{vol}(S^2) + \frac{a(A) dt dr}{r^2} \dots$

--- ~~will~~ have term $\theta \int F \wedge F$. $a_{cl} = \frac{i e^2}{4 \pi} \left(n - \frac{a_B}{2 \pi} \right)$
 With a relax.

\Rightarrow ~~co~~ ~~corr~~ correlations in presence of magnetic fields depend on G -angle.

Nonabelian case : $F = \frac{B}{2} \text{vol}(S^2) + \dots$ negative weight
 B section of adjoint bundle

$$\exists \beta : U(1) \rightarrow G \text{ homomorphism} \quad e^{2\pi i \beta} = \frac{1}{G}$$

\Rightarrow can rotate B to a fixed Cartan subalgebra

$\hookrightarrow B$ belongs to lattice in Cartan

s.t. $\alpha(B) \in \mathbb{Z}$ & any root --- i.e. B is a ^{closed} _{weight} lattice

weight lattice / Weyl group

irrep of Lie algebra's dual

Nonabelian $SU(2)$: in abelian case have just
two integers $\binom{n}{m}$, $\binom{a b}{c d} \binom{n}{m}$ $SU(2)$ _{act}

Dyonic: have rep of G , rep of G^r

$\theta \rightarrow \theta + 2\pi$ should shift electric part by magnetic
charge: $n \mapsto n+m$, $m \mapsto m$.

$\dots \Rightarrow$ but can't take max of gray orbits.

Nonzero magnetic charge: gauge field breaks down some
symmetry near boundary. have now only of
unbroken piece of symmetry...

B breaks W symmetry \Rightarrow really ^{non-triv} weight modulo unbroken W , reflection

$w_a : \alpha(B) = 0$

So physical answer for dyonic charge (μ, B)
up to unbroken W on μ weight weight

$\hookrightarrow (\Lambda \oplus \Lambda^r)/W$ [set $SU(2)$ act?]

ST of $SU(2)$ really cut here: well for simply
laced Lie algebras...

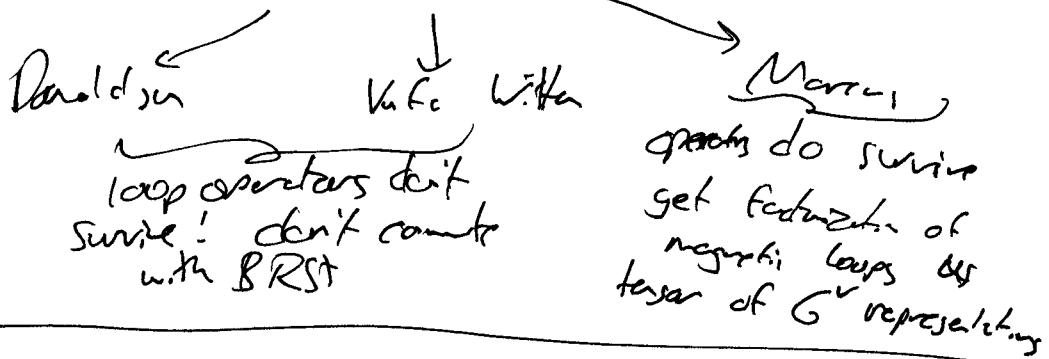
$\tilde{\Gamma} : (\mu, B) \mapsto (\mu + B, B)$ (fix instant form)

$S: (M, B) \mapsto (-B, M)$ not well defined:
 μ doesn't lie in coisotropic lattice...
so set a subgroup of $SL_2(\mathbb{R})$ as duality group

$$S T^g S \quad g = \begin{cases} 1 & F_4 \\ -1 & G_2 \end{cases}$$

$\Leftrightarrow \Gamma_0(g)$ action.

Topological twists of $N=4$ symm



Symmetry groups (Hecke groups): generated by $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$
& $(\begin{smallmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{smallmatrix})$

W sharper angled modular domain

F_4, G_2 Hitchin fibers not self-dual!
duality acts nontrivially on base
 G_2 : degree 6 Casimir gets reflected...
... have to act nontrivially on Higgs field.

Funkya category vs deformation quantiz.:

(M, ω) symplectic & complex $\omega = \text{Re } \Omega$ Ω holomorphic symplectic
 \Rightarrow take $f = \text{Im } \Omega$ form $(0,1) \oplus (1,0)$

got bisection, can deform $f = \text{Im } \Omega$ & take
modules over the bisection quantizer
e.g. T^*M M complex ...

need points of f to be discretized

(X, ω) symplectic, $Y \subset X$ ~~foliated~~ foliated by
 $\ker \omega|_Y = L$ has 2-form $F \in \Omega^2_{\text{cl}}(Y)$ [“curvature”]
 with $d_X F = 0 \quad \forall X \in L \quad [\Rightarrow F \text{ basic}]$
 & let $\sigma = \text{descat of } \omega \text{ to 2-form in transv}$
 directions, as does F
 \Rightarrow 2 form f, σ or square of (σ, ω)
 $(\sigma^{-1} f)^2 = -1$, A-C structure on leaf space
 in fact this is automatically integrable,
 \Rightarrow leaves form holomorphic symplectic manifold.

F! ~~TM~~ modulator Higher rank case now?

Canonical A-brane! doesn't usually exist, but when
 we have some fully isotropic brane

$$\Rightarrow (\omega^{-1} F)^2 = -1$$

$$\Rightarrow \text{hol symplectic form } \omega + iF$$

holomorphic strings beginning & ending on this brane are (locally)
 differential operators