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d-dim Topological Field theory: X manifold \rightsquigarrow abelian group
 $\text{Hom}(X, Y) := ?$ $\stackrel{?}{=} K_0(\text{additive category})$

In general:

C additive category $\xrightarrow{?}$ categorification of C :

If $E, F \in \text{Ob } C$ want to construct new additive category $\widehat{\text{Hom}}(E, F)$ s.t. $\text{Hom}_C(E, F) = K_0(\widehat{\text{Hom}})$

Conversely: decategorification:

$C^{(2)}$ 2-category, $E, F \in C^{(2)} \Rightarrow$ category $\text{Hom}_{C^{(2)}}(E, F)$ additive, triangulated etc.

$\rightsquigarrow K^0(C^{(2)})$: some objects,

$\text{Hom}_{K^0(C^{(2)})}(E, F) = K^0(\text{Hom}_{C^{(2)}}(E, F))$

Alternatively could take $K(C^{(2)})$,

Hom is K -theory spectrum of $\text{Hom}_{C^{(2)}}(E, F)$

\Rightarrow category enriched / spectra, can get triangulated version.

Example 1. (R. Meyer, R. Nest)

Category with objects = (nonunital) C^* -algebras.

$\text{Hom}(A, B) = KK(A, B)$ Kasparov (bivariant)

K -theory mixes $K^*(A)$ & $K_*(B)$.

... an abelian group (\approx K -theory of Kasparov bimodules)

In fact this is a triangulated category.

$$\text{Shift : } \otimes \begin{array}{c} \rightarrow \\[-1ex] \text{at } \infty \end{array} (R') \quad \text{Functor vanishing at } \infty,$$

$$\text{2-periodic : } [2] \xrightarrow{\sim} [0]. \quad \text{"noncommutative topology"}$$

Toy model in pure algebra:

k -field. 2-category w/ objects:
assoc. unital k -algebras

$\text{Hom}_{(2)}(A, B) =$ subcategory of $A\text{-mod-}B$
which are fin. gen &
projective as $B\text{-mod}$.

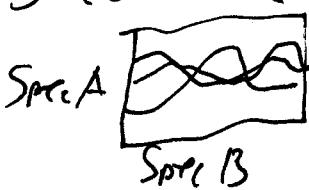
e.g. if $f: A \rightarrow B \Rightarrow$ associate graph
 $\text{graph}(f) = B$ as $A_f\text{-mod-}B$.

This is an exact category & can pass to k -theory

$\Leftrightarrow \text{Hom}_{(2)}(A, B) =$ additive functor:
(f.g. projective) $\xrightarrow{\quad}$ (f.g. projective)
 $A\text{-modules}$ $\xrightarrow{\quad}$ $B\text{-modules}$

\Leftrightarrow functor $B\text{-mod} \rightarrow A\text{-mod}$ preserving
all \varprojlim & \varinjlim \hookrightarrow having both
left & right adjoints

A, B commutative : module is graph of correspondences with stalks
finite over B , arbitrary / A .
Similar to Verdier category but
with vector bundles on correspondences.



Example 2 "Non-commutative motives"

Reminder: Grothendieck pure motives /k

$\text{Ob} = \text{smooth proj. varieties of pure dimension } [X]$

$\text{Hom}(X, Y) = \left\{ \begin{array}{l} \text{cycles in } X \times Y \text{ of dimension } \\ = \dim X \end{array} \right. \quad \left. \begin{array}{l} \text{in like graph of} \\ \text{morphisms} \end{array} \right\}$
 f.g. abelian group

Numerical
equivalence

Num. equiv: $\sum n_i Z_i = 0$ if \forall cycle Z'
 of dim $= \dim Y$, $\sum m_i (Z_i) \cap [Z'] = 0$

(believed to be \leftrightarrow homological equivalence)

Next must add several things: $[P'] \rightarrow pt \rightarrow [IP']$
 projector in this category,

formally add the direct summand

$$[IP'] = [pt] \oplus \text{Lefschetz motive}$$

Finally add Tate motive $= (\text{Lefschetz})^{-1}$
 plus its tensor powers.

In fact take Karoubi closure of all summands coming from projectors.

Could also look at \mathbb{Q} -cycles or $\overline{\mathbb{Q}}$ -cycles,
 get more & more projectives \leadsto more objects

Believed to be rigid semi-simple tensor categories.

A little bit different version: (\mathbb{Q} -coefficients).

$$\left\{ \text{pure motives} / \text{action of } \mathbb{Z} \quad \begin{matrix} \text{quotient} \\ X \rightarrow X(n) \end{matrix} \right. \quad \left. \begin{matrix} \text{Tate twist} \\ \text{in Tate twist} \end{matrix} \right.$$

$$\text{Hom quotient } ([X][Y]) = \frac{\text{cycles of all possible dimensions in } X \times Y}{\text{(Kronecker completion)}} \otimes \mathbb{Q}$$

Give another description: first look at cycles mod rational equivalence

$$(\text{cycles in } X \times Y / \text{rat. equiv}) \otimes \mathbb{Q}$$

$$\uparrow \downarrow \text{ (Lem character)}$$

$$K^0(D^b(\text{Coh}(X \times Y))) \otimes \mathbb{Q}$$

\Rightarrow 2-category: \mathcal{O}_S = smooth proj. scheme/ k

$$\text{Hom}_{\mathcal{O}_S}(X, Y) = \text{triangulated category } D^b(\text{Coh}(X \times Y))$$

that of as functors
 $D^b(\text{Coh } X) \rightarrow D^b(\text{Coh } Y)$

Numerical equivalence: On $K^0(D^b(\text{Coh } X \times Y))$ have bilinear form with values in \mathbb{Z}

$$\langle [\mathcal{E}], [\mathcal{F}] \rangle = \chi(R\text{Hom}(\mathcal{E}, \mathcal{F}))$$

Enter characteristic

Left kernel of this form = right kernel
 (even though not symmetric). Use Sone functor

$$R\text{Hom}(\mathcal{E}, \mathcal{F})^* = R\text{Hom}(\mathcal{F}, \mathcal{E} \otimes K_{X \times Y}[\text{dim } X \times Y])$$

$\xleftarrow{\text{Ch}}$ Kernel of numerical equivalence
 (up to Todd class)

Larger class of spaces \supset smooth project varieties
 "Noncommutative smooth
 proper varieties"

Definition: A unital dg algebra

$$\text{"proper": } \sum \text{rk } H^i(A) < \infty$$

"smooth": look in dg category of dg bimodules,
 its H^0 is triangulated (= derived category
 of A -bimodules). Smoothness means:
 look at S -modules $A, A \otimes A$.

A itself is a direct sum of finite
 extensions of $A \otimes A[n]$

... look at smallest ~~thick~~^{triangulated} category containing $A \otimes A$
 \hookrightarrow thick subcategory gen by $A \otimes A$

~~Smoothness~~: this subcategory should contain
 diagonal bimodule A .

(Theorem: M. Vanden Bergh & A. Bondal, explaining
 Thomason: any scheme \longleftrightarrow dg algebra)

These smooth proper A are our objects

Morphism: category of $A\text{-mod-}B$ dg bimodules
 with finite dim total cohomology.

\rightarrow "NC motives", includes quiver algebras,
 NC projective spaces etc.

NC mixed motives triangulated category.

Stay with same class of algebras. Instead of
 K^0 take K-theory spectrum of category
 of fin. d.m. dg bimodules $A\text{-mod-}B$

What does this mean: for finite simplicial complex X describe maps into this K-theory spectrum.

Approach: replace each simplex by \mathbb{A}^n ,
glue together get singular affine scheme \mathbb{Z}, X^{alg} .

Look at category of perfect complexes on X^{alg} .

Can now look at $\text{Fund}_\bullet(\text{Perf}(X^{alg}))$, $A\text{-mod-}\mathbb{B}$
now mod out by homotopy equivalences
 \rightsquigarrow describes K-theory type of this spectrum.
 \top

So get category enriched over spectra.
 Add cones & take Karoubi envelope
 \Rightarrow NC mixed motives.

Thus X, Y smooth proj varieties

$$R\text{Hom}_{\text{NCMMot}}([X], [Y]) = K_*(X \times Y) \otimes \mathbb{Q}$$

K-theory of product

In usual category of mixed motives, we use
Adams grading on $K_* \otimes \mathbb{Q}$,

$$K_*(X \times Y) \otimes \mathbb{Q} = \bigoplus K_{\mathbb{Q}}^{ij}(X \times Y) \quad \begin{matrix} \text{circles of} \\ \text{Adams grading} \end{matrix}$$

As $\mathbb{Z}/2$ graded vector space get same from this Adams as before. but \mathbb{Z} grading is slightly different

What are cohomology theories for NC pure motives?

Étale cohomology: $H_{\text{ét}}(X \times \text{Spec } \bar{k}, \mathbb{Q}_\ell)$ as Galois module
doesn't work: we've quotiented out
 by Tate motives $\mathbb{Q}(1) \sim \mathbb{Q}(0)$
 so don't get Galois module.
 (problem: use cycles on $X \times Y$ of arbitrary dimensions)
 Turns out that can't generalize (as far as we know) pure-dimensional version, which keeps Tate motives around.

Instead introduce ∞ -dim Galois module

$L = \bigoplus_{n \in \mathbb{Z}} \mathbb{Q}_\ell(n)$ or rather a Laurent completion
 of it $(\mathbb{H}^\bullet(\mathbb{P}^\infty))_{\mathbb{Q}_\ell}$
~~is~~ commutative algebra in Galois modules.

Guess: NC étale cohomology should be

$$H_{\text{ét}}^{\text{non}}(X) := H_{\text{ét}}(X) \otimes_{\mathbb{Q}_\ell} L = \text{free finite rank } L_{\mathbb{Q}_\ell}$$

For NC pure motives expect to get a free L -module
~~with~~ with Galois module, with tensor product being \otimes_L .

If $\text{Char } k=0$ can make de Rham cohomology: reduced
 A dg algebra $\rightsquigarrow C_*(A, A)$ Hochschild complex
 $= \bigoplus_{n \geq 0} A \otimes (A/k[1])^{\otimes n}$

$$\begin{matrix} b & +1 \\ B & -1 \end{matrix}$$

$$\text{If } A = \mathcal{O}(X) \Rightarrow (\mathbb{H}^*(C(A), b), B) = (\Omega^*, d)$$

X smooth affine

Negative cyclic complex: $C_-(A) = C(A, A)[[u]]$, $b+uB$
 $\deg u = +2$.

$$C_{-per}(A) = C(A, A)((u))$$

If A describes alg. variety X then

$H^*(C_{-per}(A))$ is a finite rank module/ $k((u))$,
 \mathbb{Z} -graded.

$$\deg 0 : H^0(C_{-per}) = H_{dR}^{\text{even}}(X)$$

$$H^1(C_{-per}) = H_{dR}^{\text{odd}}(X)$$

... again get bundle over Laurent series in u
 (like old expectation)

& Gal action \longleftrightarrow Gal action (\mathbb{Z} -grading)

Corfu: Hodge dR

Suppose have \otimes dg category (symmetric monoidal)
 ... e.g. usual commutative varieties.

Maybe with this extra structure can define
 Adams operators? (they break down for
 non \otimes equivalence)

Expect NC Gathen-like motives \longleftrightarrow commutative motives.

Example 3 ... ? higher-dimensional Langlands
 correspondence? ($\mathrm{char} > 0$)

(Remark: k -field \leadsto Mot(k))

(1)-line category of pure motives, conj. semi-simplicy

$k = \mathbb{F}_2 \Rightarrow$ Milne (using Tate conjecture)
gives complete description of $\text{Mot}(\ell)$:

Simple objects = simple objects in a category

$$\left\{ \begin{array}{l} V \text{ vector space over } \mathbb{Q}, \text{ F.d.} \\ F : V \rightarrow V \text{ semisimple,} \\ \det(1 - tF) \in \mathbb{Z}\left[\frac{1}{q}\right][t] \\ \text{all eigenvalues have norm } \in q^{\mathbb{Z}/2} \end{array} \right\}$$

$\otimes \bar{\mathbb{Q}}$ this category is the same as $\text{Mot}(\ell) \otimes \bar{\mathbb{Q}}$
over \mathbb{Q} they're different.

Langlands conjecture: gives conjectural description of
motives of 1-dim fields, e.g.

$$k = \mathbb{F}_q(C) \quad C \text{ curve.}$$

Simple motives of $rt = N$ with coeffs in $\bar{\mathbb{Q}}$

? = cuspidal automorphic representations of $GL_n \mathbb{A}_k$.

$C = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ Want to illustrate meaning of these words
in simplest example

$\text{Irrep} \left(\pi_{\text{geom}}^{\text{geom}} (C \setminus \{0, 1, \infty\}) \xrightarrow{\text{continuous}} GL_2 \bar{\mathbb{Q}}_l \right) / (\text{conj.})$
tame, unramified
dimensional ramifications $(\ell, q) = 1$

$\hat{\mathbb{Z}} = \text{Gal}(\bar{\mathbb{F}}_q / \mathbb{F}_q) \rightarrow \text{Aut } \pi_{\text{geom}}$

acts on this space,

consider $(\text{Irrep})^F$ invariant

Get set with $\text{Gal}(\bar{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ action
(in fact finite set)

\iff look at arithmetic Π_1 , which maps
to $\hat{\mathbb{Z}}$, so ~~then~~ can tensor Π_1 with $\text{mod-}\ell$,
by $\hat{\mathbb{Z}}$ mod- ℓ s. Above problem:
look at geom. mod. reps of Π_1 with
up to translation by $\hat{\mathbb{Z}}$ mod- ℓ s

Langlands tells us how to get an automorphic
description of this set, which is elementary

Langlands: Galois spectrum = Hecke spectrum.

Look at Hecke spectrum on automorphic forms
- here functions on finite sets.

Take Hecke operators only at \mathbb{F}_q -points (be
(rank) of bulk is 1-dim here, so this is enough)

Space of Aut forms: functions $\mathbb{F}_q \rightarrow \mathbb{C}$

Hecke operators $T_x \in \text{Mat}(\mathbb{F}_q \times \mathbb{F}_q, \mathbb{Z})$

Matrix coeffs for $y, z \in \mathbb{F}_q$:

$$(T_x)_{y,z} = 2 - \#\{w \in \mathbb{F}_q : w^2 = f_x(x, y, z)\}$$

$$- \begin{cases} q+1 & x=y \in \{0, b, t\} \\ 1 & x=y \notin \{0, b, t\} \end{cases}$$

$$+ 2q \underbrace{\left(\frac{x}{b} + \frac{t}{b} \right)}_{\text{if } x \neq t} \underbrace{\left(\frac{y}{x} + \frac{z}{x} \right)}_{\text{if } y \neq z}$$

$$+ \begin{cases} q & \text{if } x \notin \{0, 1, t\} \text{ &} \\ 0 & \text{otherwise} \end{cases} \begin{cases} y = \frac{t}{x} & z=0 \\ y = \frac{t-x}{1-x} & z=1 \\ y = \frac{t(1-x)}{t-x} & z=t \end{cases}$$

where $f_t(x, y, z) = (xy + yz + zx - t)^2$

$$+ 4xyz(1+t - (x+y+z))$$

(common eigenvalues of these matrices (T_x)
 (which commute and are semisimple)
 \longleftrightarrow set of irrep's before.

$q=101$ char. polynomial decomposes into product
 of 4 polynomials of order 25, 25, 25, 25.

-- because T_0, T_1, T_t are involutions (Fröbenius inv'tns)
 $\{T_0, T_1, T_t, \mathbb{I}\} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

& In this case $T_x \cdot T_y = \sum c_{xyz} \cdot T_z$ close
 & in fact $c_{xyz} = T_{x,y,z}$ (matrix entry)

Observation: these calculations over \mathbb{F}_q
 have a simple structure, with answer notivic,
 no finite fields involved.

Guess: Jacquet-Langlands Automorphic forms for GL_2 / Functor Circle
 has description independent of fields

Formalism k any field
(poor man's version: dequantization).

Category \mathcal{E}_k : $Ob =$ constructible sets /k (subset of $A^{n!}$)

$Hom(X, Y) =$ formal linear combinations of
constructible sets $Z \rightarrow X \times Y$

$$\text{with } Z = Z_1 \cup Z_2 \quad [Z]_{X \times Y} = [Z_1]_{X \times Y} + [Z_2]_{X \times Y}$$

(Drezet) Define motivic numbers, & motivic functions
(poor man's version)

Grothendieck group of varieties

"Motivic number": $[X]$ variety /k s.t. if $Y \subset X$ closed

$[X] = [Y] + [X - Y]$. Call its elements
motivic numbers. If k is finite
can get morphism to \mathbb{Z} by counting points.

Motivic function S algebraic variety

$\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ varieties /S. Play same game rd S
 \Rightarrow Motivic functions on S.

Analog for k finite motivic functions
give usual functions. . Form a ring

Maxim's version : $Hom(X, Y) =$ motivic function on
composition = matrix multiplication $X \times Y$

Target-languages for function fields can be rewritten in this language.

To get sophisticated notations: look at formalism of varieties + morphisms.

(Kontext) This category is a rigid \otimes category,
 $[X] \otimes [Y] = [X \times Y]$.

Rigid: all $X \cong X^*$ where $[X \times X] \xrightarrow{\text{pt}} \mathbb{I}$
 is just $\text{diag}_X \in X \times X \times \text{pt}$.

$k = \mathbb{F}_q$ $\forall n \geq 1 \Rightarrow$ tensor functor

$\varphi_n : C_{\mathbb{F}_q} \longrightarrow$ fin dim vector spaces / \mathbb{Q}

$X \longmapsto \mathbb{Q}^{X(\mathbb{F}_{q^n})} = \mathbb{Q}\text{-valued functions}$
 on $X(\mathbb{F}_{q^n})$.

$\begin{matrix} Z \\ \downarrow \\ X \times Y \end{matrix} \longmapsto \text{matrix } \varphi_n \left(\begin{smallmatrix} Z \\ X \times Y \end{smallmatrix} \right) (Y, Y)$
 $= \# \left\{ z \in Z(\mathbb{F}_{q^n}) \mid \begin{smallmatrix} Z \\ (x, y) \end{smallmatrix} \right\}$

Proof of above formulas in fact works in this category.

Conclusion of above formulas: get
 a comm. assoc. algebra in $C_{\mathbb{F}_q}$
 = global Hecke algebra, gives
 concrete algebras by applying φ_n 's.

Langlands $\left\{ \text{irreps } \pi_i^{\text{gen}}(C_{X,x}) \rightarrow GL_N(\bar{\mathbb{Q}_\ell}) \right\}$
 with bounds on ramification (conjecture)

local at F_n^n fixed mts

(conjecture) $\stackrel{V_n \mathbb{Z}^\times}{\{ \text{Irr } \pi_i \rightarrow GL_N \}} = ? \text{Hom}(\varphi_n(A), \bar{\mathbb{Q}})$

where A is a commutative algebra in C_{F_ℓ}
 depending on C, N
 (compatible with $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{Z}/n\mathbb{Z}$ symmetry)

(Dinfd) Langlands conjectures (for GL_N & a curve X/k or X/S)

\Rightarrow algebra $A \in C_S$, canonically defined.
 easy to define, commutative.

Main property: if $S = \text{Spec } F_2$

& apply φ_n to A , then $\text{Spec } \varphi_n(A)$
 bijectively correspond to Frob fixed mts
 in ~~rep~~ representation.

Key features : $\begin{array}{l} \bullet \exists A \\ \bullet A \text{ very explicitly defined} \end{array}$

Kontsevich conjectures can still define such A (commutative)
 for any dimension variety instead of a curve.

Describe A ~~very~~ explicitly by structure constants $c(ijk)$
 $i, j, k \in X$ variety whose motive is A

(an also up to change of bases: Kontsevich doesn't give a preferred basis.)

(Kontsevich) Why to believe this? e.g. S projective surface,

$C \subset S$ curve, ample $\Rightarrow \pi_1(C) \rightarrow \pi_1(S)$
 so there are fewer local systems on S .
 expect quotient algebra of curve \mathcal{A} .

(Drinfeld) Open problem: suppose C, S over \mathbb{F}_q .

$\pi_1(C) \rightarrow GL_2(\mathbb{Q}_\ell)$ ℓ -adic reps

have action of compatible families of such:
 all eigenvalues of Frobenius are only integers $(\bmod \ell^{k_m} \alpha_m)$
 So if have ℓ -adic rep ~~but~~ ℓ -adic reps
 can ask: if there are ℓ^r -adic reps with
 same eigenvalues? A: Yes (follows
 from Langlands description: have ~~too~~ same
 automorphic spectrum, over \mathbb{Q} .)

Given $\pi_1(C) \rightarrow \pi_1(S)$

$$\begin{array}{ccc} & \nearrow & \downarrow \\ GL_2(\mathbb{Q}_\ell) & & GL_2(\mathbb{Q}_\ell) \end{array}$$

Does this compatible rep also factor through
 $\pi_1(S)$? ??

(would follow from Kontsevich conjecture)

(Kontsevich). $p_1, p_2 : H^{\text{top}}(S) \rightarrow GL_2(\mathbb{Q}_\ell)$

can calculate Ext groups!

$H^i_{\text{et}}(S, \rho_1^* \otimes \rho_2)$ over $\bar{\mathbb{Q}_\ell}$:

Dimensions of these jump on Brill-Noether loci.

Conjecture they come from bimodules in $C_{\mathbb{F}_q}$

Richman's versions form 2-category,

X, Y schemes, $\text{Hom}(X, Y) = \text{constructible motivic sheaves } / X \times Y$
above should come by taking K_0 .

(Ext conjecture above close to Lazard's functoriality:

$A^{(n)}$ describes n -dim reps of $\pi_1^{\text{geo}}(S)$

$A^{(m)}$ mod. n rep except Ext
to relate to $A^{(n)} - \text{mod} \not\cong A^{(m)}$

Example 4: Lattice models.

$X \in \text{Ob } C_{\mathbb{F}_q}$, $M \in \text{End}_{C_{\mathbb{F}_q}}(X)$ motivic function
in 2 variables

\Rightarrow family of finite matrices $\varphi_n(M)$ $\forall n \geq 1$,

with more & more complicated spectrum. What is it??

Understand $\text{Spec } \varphi_n(M) \subset \mathbb{C}$ as $n \rightarrow \infty$

... study char. polynomials $(\text{size } \otimes X(\mathbb{F}_{q^n})^2)$

$Z(n, m) := \text{Trace}(\varphi_n(M)^m)$ $n, m \geq 1$

$m \geq 1$: ignore zero eigenvalues of $\varphi_n(M)$

Observations

1. Fix $n \Rightarrow \exists$ finite collection $\{\lambda_\alpha\} \subset \mathbb{C}$
 (obvious) s.t. $Z(n,n) = \sum \lambda_\alpha^n \quad \forall n$

Weil conjecture) 2. Fix $m \Rightarrow \exists$ finite collection $\{\mu_\alpha\} \subset \mathbb{C}^*$
 $\epsilon_\alpha = \pm 1$ s.t. $Z(n,n) = \sum \epsilon_\alpha \mu_\alpha^n \quad \forall n$.

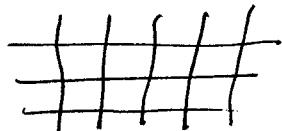
Why? $\text{Trace } q_n(M)^m = \text{Trace } (q_n(M^m))$

i.e. have variety $Y_{(m)} \rightarrow X \times X$ corresponds to M^m .

$\text{Trace} = \#(Y_{(m)} \cap \text{Diag}) (\mathbb{F}_{q^n})$
 = trace of Frobenius on cohomology

- so behavior in m, n symmetric (both cases
 ~~$\# \lambda, \# \mu$ grows exponentially)~~)

2-dim lattice models

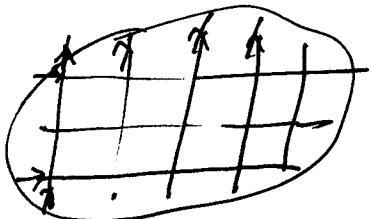


$V_1, V_2 \in \text{Vect}/\mathbb{C} \quad \dim V_i < \infty$

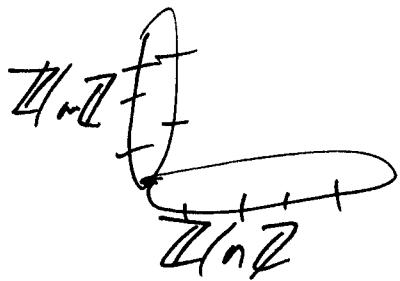
$R : V_1 \otimes V_2 \rightarrow V_1 \otimes V_2$. Want to calculate traces
 of powers of R .

Choose bases of $V_i \Rightarrow$ tensor with 4 indices

$\rightarrow \begin{matrix} i' \\ j \end{matrix}, \quad i, i' \in \text{basis of } V_1 \quad j, j' \in \text{basis of } V_2$

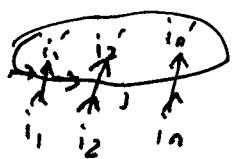


Fix boundary values of indices,
 take sum over all ways to complete
 inside.



look in doubly periodic setting (boundary conditions), calculate F_{Γ} ,
partition function get
function of $\eta_{\alpha m} = \underline{\text{trace}}(T_{(n)}^m)$

Operator $T_{(n)}: V_i^{\otimes n} \rightarrow V_i^{\otimes n}$



with matrix coeffs

$$T_{(n)}^{i_1 \dots i_n} = \sum_{j_1, \dots, j_n = j_0} \prod R_{i_n \dots i_1}^{j_n \dots j_1}$$

(transfer matrix)

If V_i super vector spaces R even \Rightarrow
gives as above but with signs.

Q: Given $X \in C_{\mathbb{F}_q}$, $M \in \text{End}(X)$

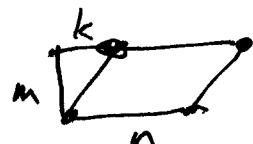
does there exist a lattice model producing the same number?

Given a lattice model could also formulate for
 $\forall \Gamma \subset \mathbb{Z}^{n,2}$ of finite index,

get $Z(\Gamma) \subset \mathbb{C}$

\therefore lattice Γ depends

on three parameters $n, m > 1$, $k \in \mathbb{Z}/n\mathbb{Z}$.



Geometric version: $\text{Tr} (q_n(M)^m \cdot q_n(F_{r_X})^k)$

F_{r_X} = Frobenius operator on X .

Claim: In both cases, $\forall v_1, v_2 \in \mathbb{Z}^2$ noncollinear ($v_1 \wedge v_2 \neq 0$) \exists finite collection λ_α & signs \pm_α s.t $\forall N \geq 1 \quad \mathbb{Z}(\mathbb{Z}v_1 \oplus N \cdot \mathbb{Z}v_2) = \sum \pm_\alpha \lambda_\alpha^N$.
 (generalization of above property for n, n')

Say two lattice models are isospectral if same λ_α 's. $SL_2 \mathbb{Z}$ acts on isospectral lattices up to isospectral equivalence, can reduce question to simpler lattices...

d-dim lattice model, $d \geq 0$:

$$V_1, \dots, V_d \in \text{Vect}'/\mathbb{C} \quad \text{dim } V_i < \infty$$

$$R: V_1 \otimes \dots \otimes V_d \hookrightarrow$$

\Rightarrow partition function on all finite index lattices $\Gamma \subset \mathbb{Z}^d$.

Similar behavior of λ_α 's ...

--- can formulate such in any rigid tensor category
 (partition function will be endomorphism of identity object)

--- in particular in $\mathcal{G}_{\mathbb{F}_q}$.

Q: Given a d-dimensional lattice model M_d in $\mathcal{G}_{\mathbb{F}_q}$, does there exist a $(d+1)$ -dim numerical lattice model M_{d+1}^{num} s.t.

$$\forall n \geq 1 \quad \varphi_n(M_d) \sim \begin{array}{l} \text{Kazakov-Klein reduction} \\ \text{isospectral} \end{array} \quad \mathbb{Z}/n\mathbb{Z} \quad \text{d-dim}$$

$d=0$ case: need morphism $\bullet \rightarrow \bullet$, ie
a variety, and we're calculating its number
of points over finite fields,
statement follows from Weil conjecture.

Hecke operators: counting operators \longleftrightarrow
counting transfer matrix (R-matrices)
depending on variable z , with counting coefficients

Consider 2-category ~~A~~,

Obj = f.d. supervector spaces / \mathbb{C}

Item ~~A~~ $(U, V) =$ finite dim ^{super} reps of tensor alg^{re}
 \otimes^* $(U \otimes V^*)$ like map
 $\text{Rep}(A)$

i.e. map $U \otimes V^* \otimes E \rightarrow \mathbb{E}$
 $\Leftrightarrow U \otimes E \rightarrow V \otimes E$.

Composition $V \otimes F \rightarrow W \otimes F$

$\Rightarrow U \otimes E \otimes F \rightarrow W \otimes E \otimes F$

b: additive functor. ~~A~~ is a \otimes -2-category.

$\varphi_n: K_0(\mathcal{A}) \rightarrow$ f.d. Vect (apply K_0 to
hom categories)

$V \mapsto V^{\otimes n} \hookrightarrow \mathbb{Z}^{n \times n}$

Given morphism $U \otimes E \rightarrow V \otimes E$ $\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & \nearrow g & \downarrow h \\ E & \xrightarrow{f'} & E \end{array}$

can build $\begin{array}{c} \xrightarrow{U} \xrightarrow{f} \xrightarrow{V} \\ \downarrow & \nearrow g & \downarrow h \\ \xrightarrow{E} \xrightarrow{f'} \xrightarrow{E} \end{array}$ like before

Frobenius $\text{Fr}_V: V \otimes V \rightarrow V \otimes V$ for $V=V$
 action of permutation.

On $V^{\otimes n}$ get action of cyclic permutn

Conjecture \exists \otimes functor $(\mathbb{F}_q \rightarrow K_0(\mathbb{A}))$
 (& categorizn)

$$\begin{array}{ccc} & \otimes & \\ \mathbb{F}_q & \downarrow \varphi_n & \downarrow \varphi_n \\ & \text{Vect} & \end{array}$$

..... implies lattice model conjecture above.

② Would give canonical construction of finite fields:

Nice object $A' \in \mathcal{C}_k$, which is
 a commutative algebra (multiplication given
 by diagonal $\Delta_3 \subset (A')^3$).

... in fact this is a ring scheme object in \mathcal{C}_k .

(ie $A' \otimes A' \rightarrow A'$ & two coproducts $A' \rightarrow A' \otimes A'$)

Now if above functor exists, apply it to
 A' , get vector space C^α , with
 operations satisfying some rules ...

on $\varphi_n(A')$ get \mathbb{F}^{2^n} with commutative alg - strucy
 simulates structure of finite field \mathbb{F}_{q^n} ...

Carrying in addition of digits
 is 2-d lattice mod!