S. Loktev – Current Algebras & Representations 10/27/04
(w. B. Feigin) $\mathfrak{b} \subset \mathfrak{g}$ reductive. A assoc algebra w/unit $= \Gamma(M, \mathcal{O})$ M affine

~~Def~~ Def. (following Chari-Pressley $\to$ 1-dim M case)
construct representations of current algebra $\mathfrak{g} \otimes A$
$\lambda : \mathfrak{b} \to \mathfrak{h} \to \mathbb{C}$ $\quad \varepsilon : A \to \mathbb{C}$ augmentation

Def Weyl module $W_{\lambda,\varepsilon}$ = Maximal fin dim $\mathfrak{g} \otimes A$-module gen. by vector $v_{\lambda,\varepsilon}$

$$(g \otimes P)\, v_{\lambda,\varepsilon} = \lambda(g)\, \varepsilon(P)\, v_{\lambda,\varepsilon}$$

for $g \in \mathfrak{b}$ $\quad P \in A$

[Maximal : maps to any such module – maximal fin dim quotient of induced module, if it exists

Theorem (Chari-Pressley in smooth 1-dim case)
1. $W_{\lambda,\varepsilon}$ exist
2. $\mathfrak{g} \otimes A_{\varepsilon}^{N}$ acts by zero for $N \gg 0$
3. Any fin. dim. $\mathfrak{g} \otimes A$ module generated by a common eigenvalue of $\mathfrak{b} \otimes A$ Borel valued currents is a quotient of a product of $W_{\lambda_i \varepsilon_i}$

Higher Weyl modules:

Definition $W_{\lambda,\varepsilon}^{(k)}$ = submodule of $W_{\lambda,\varepsilon}^{\otimes k}$ generated by $v_{\lambda,\varepsilon}^{\otimes k} =: v_{\lambda,\varepsilon}^{(k)}$

Algebraic structure: exist map

$$W_{\lambda,\varepsilon}^{(k_1)} \otimes W_{\lambda,\varepsilon}^{(k_2)} \leftarrow W_{\lambda,\varepsilon}^{(k_1+k_2)} \qquad v_{\lambda,\varepsilon}^{(k_1)} \otimes v_{\lambda,\varepsilon}^{(k_2)} \leftarrow\!\shortmid\, v_{\lambda,\varepsilon}^{(k_1+k_2)}$$

-- coalgebra structure ... coassociative & cocommutative

So get structure of algebra on $\bigoplus_k (W_{\lambda,\mathcal{E}}^{(k)})^*$.
associative & commutative

If $A = \mathbb{C}$ $\Rightarrow$ Functions on Affine Schubert G(!).

Define Schubert variety $Sch_{\lambda,\mathcal{E}} = Proj\left(\bigoplus_k (W_{\lambda,\mathcal{E}}^{(k)})^*\right)$

~~6.~~ • Dim & character of $W_{\lambda,\mathcal{E}}^{(k)}$?

have an action of $\mathfrak{g}$ & some endos of ring A, preserving $\mathcal{E}$ ⇢ i.e. a certain GL (in smooth case)

Qs: 1. What is $ch\, W_{\lambda,\mathcal{E}}^{(k)}$ as $\mathfrak{g} \otimes Aut_{\mathcal{E}} A$ module?

2. Describe $Sch_{\lambda,\mathcal{E}}$

3 $Hom_{\mathfrak{g}\otimes A}(W_{\lambda,\mathcal{E}}^{(k_1)}, W_{\lambda_2,\mathcal{E}}^{(k_2)})$ & limits as $\lambda \to \infty$..

<u>Examples</u> $A = \mathbb{C}[x^1, \dots x^d]$ $\mathcal{E}(P) = P(0)$

• $d=0$: $W_{\lambda,\mathcal{E}}^{(k)} = \pi(k\lambda)$ irreducible representation of $\mathfrak{g}$

Coalgebra is well known. $Sch_{\lambda,\mathcal{E}}$ = image of G/B in $\mathbb{P}(\pi(\lambda))$

no nontrivial Homs ...

• $d=1$ Demazure modules.
Consider $\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}c$ central extension

$L_{k,\lambda}$ integrable representations level k.

[Statement ~~(known for $\mathfrak{gl}_n$)~~]

$v \in L_{k,\lambda}$ highest weight vector. Weyl group acts on $L_{k,\lambda}$

$v_w = w \cdot v$ $w \in W$,

$V_\alpha = T_\alpha \cdot V \qquad \alpha \in T \subset W$

Demazure $\quad D^{(k)}_{\lambda + k\alpha} = U\mathfrak{g} \otimes \mathbb{C}[t] \cdot V_\alpha$

rep of $\mathfrak{g} \otimes \mathbb{C}[t] \subset \hat{\mathfrak{g}}$

<u>Proposition</u> $D^{(k)}_{\mu}$ is a quotient of $W^{(k)}_{\mu, \varepsilon}$ $\qquad$ μ dominant

... easy for $k=1$, since V_α in this case will satisfy Weyl condition..

<u>Conjecture</u> (Theorem for $\mathfrak{gl}_r$) $\quad W^{(k)}_{\mu,\varepsilon} \cong D^{(k)}_{\mu}$
as $\mathfrak{g} \otimes \mathbb{C}[t]$-modules

So Sch are affine Schubert varieties: μ in root lattice get Schubert cell in Grassmannian ... $\mathcal{O}_\mu \subset Gr$
Otherwise find it in some flag variety $\qquad G(\mathcal{O})$ orbit

<u>Corollary</u> $\mu \in Q \qquad Sch_{\lambda,\varepsilon}$ = Schubert variety in the affine Grassmannian $\quad \mathfrak{g} \otimes \mathbb{C}[[t]]$-orbit.

functions on orbit closure are always Demazure modules -- nontrivial statement is that Demazure modules are Weyl, i.e <u>maximal</u> of this given type.

Singular curve case: nodal case: T. Kuwabara ($\mathfrak{gl}_n$)

$\boxed{d=2}$ $\quad \mathfrak{g} = \mathfrak{gl}_r$

Consider noncommutative polynomials

$$\mathbb{C}\langle X, Y\rangle = \langle X, Y\rangle / \; YX - XY = X$$

(natural representation: $\quad X = x \quad Y = x\,\partial/\partial x$

(want maximal ideal to get started --- that's why $YX - XY = X$

Study $\mathfrak{gl}_r \otimes \mathbb{C}\langle X, Y\rangle$

$\mathbb{C}\langle X,Y\rangle \curvearrowright \mathbb{C}[x]$, increasing degrees of polynomials

Consider $\mathbb{C}[x^{-1},x]/x^N\mathbb{C}[x]$.

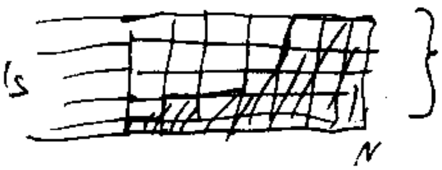
$V=\langle v_1,\dots v_r\rangle$ vector rep of $\mathfrak{gl}_r$.

ξ partition. $\quad \Lambda^{|\xi|}\left(V\otimes \mathbb{C}[x^{-1},x]/x^N\mathbb{C}[x]\right)$

<u>Def</u>. $\quad V_\xi(N) \subset$: submodule generated by

$$v_\xi = \bigwedge_{i=1}^{r} \bigwedge_{j=N-\xi_i}^{N-1} v_i\otimes x^j$$

Variables are cells inside diagram mod



Filtration by degree of differential operators:

$F^i\,\mathbb{C}\langle X,Y\rangle$, filtration on enveloping algebra of $\mathfrak{gl}_r$

$\Rightarrow$ filtration on any cyclic module

$$F^i V_\xi(N) = F^i\, U(\mathfrak{gl}_r\langle X,Y\rangle)\cdot v_\xi$$

$$\mathrm{gr}\,(\mathfrak{gl}_r\langle X,Y\rangle) = \mathfrak{gl}_r\otimes\mathbb{C}[X,Y]$$

<u>Prop</u>: $\mathrm{gr}\ V_\xi(N)$ is a quotient of $W_{\xi,\mathcal{E}}$

... can produce also higher Weyls this way.

<u>Theorem</u> $\quad V_\xi(N) \simeq W_{\xi,\mathcal{E}}$

dimensions for $\mathfrak{sl}_2$: Catalan numbers (using Haiman)

$\xi = \xi, 0, 0, \dots 0$

<u>Theorem</u> (M. Haiman) $\quad \mathbb{C}[X_1,\dots X_n, Y_1,\dots Y_n]$ with Σ_n acting on both sets of variables

which is $\mathbb{C}[\dots]/\langle \mathbb{C}[\dots]^{\Sigma_n}_+\rangle$ diagonal coinvariants.

action of z, z^{-1}
$z = y - 1$
$z^{-1} = \frac{1}{y-1} = 1 + y + y^2 \ldots$

Weyl modules = limits of q-affine modules, which generically are irreducible

$V_\xi^{(t)}(N)$ submodule in $V_\xi(N)^{\otimes t}$ generated by $V_\xi^{\otimes t}$

<u>Prop</u> $V_\xi^{(t)}(N) = \Gamma(Sch_\xi, Q^t)$ → tautological quotient bundle on Grassmann

On $Gr_{|\xi|}(|\xi| \cdot r)$ $|\xi|$-planes finite dim Grassmann

Sch_ξ : a particular Schubert cell determined by diagram.

Consider End $V \otimes (\mathbb{C}[x]/x^N \mathbb{C}[x]) = \mathrm{End}\, V \otimes \mathrm{End}(\mathbb{C}[x]/x^N)$

<u>Proposition</u>
Image of $\mathfrak{gl}_r \langle X, Y \rangle \to \mathrm{End}(\,)$ is precisely $\mathrm{End}\, V \otimes B$, "parabolic subalgebra"

$\cup$
B Borel matrices preserving filtration.

---> see Schubert cells in usual Grassmann.

Passing to assoc graded spaces : dependence on N disappears

- Conjecturally gives deformation of double loop Schubert cells

Conj: $\mathrm{gr}\, V_\xi^{(t)}(N) \simeq W_\xi^{(t)}$

- so algebra of deformed Weyl module gives def:
$Sch_{\xi, \varepsilon}$ deforms to $Gr_\xi(|\xi| r)$

$G[t, t^{-1}, z, z^{-1}] / G[t, z, z^{-1}]$

<u>Example</u> $r = 2$ $\xi = (2, 0, 0 \ldots 0)$
deformed case: $Q = (x_1^2 + \ldots + x_4^2 = 0) \subset \mathbb{P}^5$ singular quadric
nondeformed case $Q^0 = (x_1^2 + x_2^2 + x_3^2 = 0) \subset \mathbb{P}^5$
- Schubert cell for double loop Grassmannian.

Level 1 toroidal algebra acts on limit $\lim V_\lambda^*$

← Kapranov - Vasserot
(Saito - Takemura - Uglov)

stabilizer of point is Taylor part