J. Lurie : GRASP - Bezout's Theorem & Nonabelian Homological Algebra 9/7/05

Derived algebraic geometry! jazz up foundations of algebraic geometry using homotopy theory.

Bezout's Theorem $C, C' \subset \mathbb{CP}^2$ smooth curves in the plane of degrees n & m

If $C \perp C'$ (meet transversely)

then $\#(C \cap C') = n \cdot m$

Reformulate: Let $[C]$, $[C']$ be fundamental classes in cohomology of $\mathbb{CP}^2$

$$\underset{n}{[C]} \cdot \underset{m}{[C']} = \underbrace{[C \cap C']}_{= \#\text{ points of intersection}}$$

If intersection nontransverse $\Rightarrow$ $\# C \cap C'$ always $< nm$

---- need to count with multiplicity.

Suppose $C, C' \subseteq \mathbb{A}^2$ affine plane with coordinates x, y.
functions on $\mathbb{A}^2 = \mathbb{C}[x,y]$ polynomials

C : defined by $f = 0$
C' : defined by $g = 0$ $\quad$ f, g polynomials

$\mathbb{C}[x,y]/(f) =$ Functions on C : remember that $f = 0$ on C.

Functions on C, C' $\underset{\text{roughly}}{\sim}$ $\mathbb{C}[x,y]/(f,g)$

divide out by both equations ---- scheme theoretic intersection ! want not just <u>pts</u> of intersection but <u>rings</u> there e.g C : line $x=0$

C' : line $y=0$

$C \cap C' \longleftrightarrow \mathbb{C}[x,y]/(x,y) \simeq \mathbb{C}$ = fns on single intersection point.

e.g C : line $x=0$

C' : parabola $x=y^2$

$C \cap C' \longleftrightarrow \mathbb{C}[x,y]/(x, x-y^2)$

$\simeq \mathbb{C}[y]/y^2$: 2-dimensional/$\mathbb{C}$,

reflecting point of tangency : multiplicity two.

In general, for plane curves

$$n \cdot m = \sum_{p \in C \cap C'} \dim_{\mathbb{C}} \Big(\underbrace{\mathcal{O}_C}_{\text{functions on } C} \otimes_{\mathcal{O}_{\mathbb{P}^2}} \underbrace{\mathcal{O}_{C'}}_{\text{functions on } C'} \Big)_p$$

Suppose C, C' varieties of dimensions a, b sitting in $\mathbb{CP}^{a+b}$ --- still expect generically $\# C \cap C' < \infty$.

$\deg C = n$, $\deg C' = m$

$\Rightarrow$ Bezout If C, C' transverse $\Rightarrow \#(C \cap C') = nm$

Suppose first C given by a single equation $f=0$ & C' has a ring of functions R.

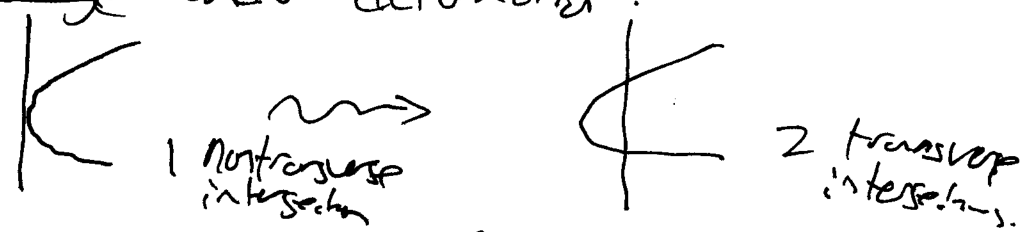
$\Rightarrow$ we should take $R/(f)$ and count dimension to find intersection multiplicity

$R \xrightarrow{f} R \to R/(f)$ cokernel of multp. by f.

Finite dim analog: $\dim R < \infty \Rightarrow$
f is a matrix, usually invertible but the rank of f can jump & dim of cokernel will change.

This is bad: we want intersection multiplicity not to change under deformation:

e.g.



Rule: • invariant under deformation/perturbation
• One for transverse intersections.

But ranks of matrices / dim of cokernel jump!

NOTE: If a matrix f has cokernel $\Rightarrow$ it has kernel
... in fact $\dim(\operatorname{coker} f) - (\dim \ker f) = 0$
... special version for R finite dimensional.

Curve case: R will be ∞ dimensional
but $\dim \ker$, $\dim \operatorname{coker} < \infty$
& get interesting number $\underbrace{\dim \operatorname{coker}}_{\text{naive approximation}} - \underbrace{\dim \ker}_{\text{perturbation}}$

$\boxed{\text{Tor}}$ A commutative ring, M, N A-modules
$\Rightarrow$ new A-modules $\mathrm{Tor}_i^A(M,N)$, with property
$\mathrm{Tor}_0^A(M,N) = M \otimes_A N$.

Other Tor_i^A are corrections, measure bad behavior of $\otimes_A$ of ...

Fancy Bezout! [Serre] C (dim a, deg n), C' (dim b, deg m) $\subseteq \mathbb{P}^{a+b}$

$\boxed{\text{assume } \# C \cap C' < \infty}$

$$\Rightarrow \quad nm = \sum_{p \in C \cap C'} \left(\sum_{i=0}^{\infty} (-1)^i \dim \mathrm{Tor}_i^{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_C, \mathcal{O}_{C'}) \right)_p$$

$i=0$ term: just dim of ring coming from imposing eqns of C & of C'.

Solves problem, but we want nice formula
$[C]\cdot[C'] = [C \cap C']$ for curves intersecting transversely, or for curves in general $\subset \mathbb{P}^2$ where $[C \cap C']$ is counted scheme-theoretically.

Scheme intersection doesn't see Tor^i corrections!
$\rightsquigarrow$ derived algebraic geometry.

Back to basics: two lines L, L' in the plane
$L \cap L'$ always one point ...... unless $L = L'$ ....
... in this case $[L]\cdot[L']$ completely different from $[L \cap L'] = [L]$!

How to correct this:

In general position can assume $L = (x=0)$, $L' = (y=0)$
intersection $\mathbb{C}[x,y]/(x,y) = \mathbb{C}$.

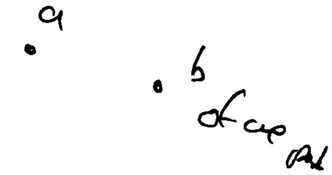
What if $L = L'$? both have $(x=0)$ as equation.

$\mathbb{C}[x,y]/(x,x) = \mathbb{C}[y]$ not finite dimensional$/\mathbb{C}$

... equations weren't independent.. imposing $X=0$ twice is same as dividing once..

Want a world where ~~intersecting twice~~ is imposing same eqn twice $\neq$ imposing once.

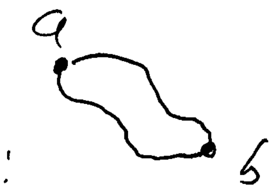
Set theory: two identify sets can just get set with one point



<u>OR</u> in topology can add a <u>path</u> from a to b:

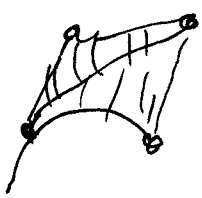
for topologists this is the <u>same</u> as identifying a, b.

BUT identifying a, b <u>twice</u> we get a different topological space!



Writing a space by generators & relations: giving identifications.

In topology have notion of CW complex: built by successively adding cells.



--- presenting a space by generators (points), relations (arcs between points), relations between relations (discs between arcs) & so on!

— now try to do same with <u>commutative rings</u> instead of sets; work with objects that are a space + commutative ring at once.

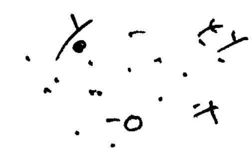
Most naive version: topological commutative rings
... topological space with continuous ring structure.

[Silly example: R any commutative ring, can give discrete topology.]

Consider R, S equivalent if have a map $R \longrightarrow S$ inducing isomorphism on all homotopy groups.

e.g. $\mathbb{R}$, $\mathbb{C}$, Banach algebras etc are all contractible $\Longrightarrow$ equivalent to zero.

Algebra: get topology from completions, e.g. $\mathbb{Z}_p$
p-adics: totally disconnected, no higher homotopy ... might as well be discrete.



Example $\mathbb{C}[x,y]$ with discrete topology

Now impose equation $y=0$ twice:
take free commutative ring generated by two paths from y to 0: start like in set theory before, but generate ring.

Naive Bezout: tensor product in wrong world...
Tors come from $\otimes$ in world of topological

$$\mathbb{C}[x,y] \xrightarrow[\substack{y\to 0\\ y\to 0}]{} R$$ topological commutative ring given by adjoining paths.

$$\pi_0 R = \mathbb{C}[x] = \mathbb{C}[x,y]/(y)$$

$\pi_1 R \ni$ canonical element ε: loop

... but also $\pi_0 R$-module

$\Rightarrow \pi_1 R = \mathbb{C}[x]\cdot\varepsilon$ & $\pi_i R = 0 \quad i>1$

Tensor product of topological commutative rings, call it $\otimes^L$ for left derived functor of $\otimes$ operation

$R = \mathcal{O}_C \overset{L}{\underset{\mathcal{O}_{par6}}{\otimes}} \mathcal{O}_{C'}$: a topological rings operation
... example of homotopical algebras.

$\pi_0 R$ connected components (ie forget topological stuff)
$= \mathcal{O}_C \underset{\mathcal{O}_{par6}}{\otimes} \mathcal{O}_{C'}$ usual tensor prod

$\pi_i R = \mathrm{Tor}_i^{\mathcal{O}_{par6}}(\mathcal{O}_C, \mathcal{O}_{C'})$ correction terms.

Suppose R topological commutative ring
$\Rightarrow$ ordinary commutative ring $\pi_0 R$, collapse all path component,
"underlying commutative ring" of R.

Other π_i : interesting "correction" information.

<u>Def</u> A <u>scheme</u> is a topological space X with a sheaf of rings $\mathcal{O}_X$ such that $(X, \mathcal{O}_X)$ locally looks like $(\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A})$

A a commutative ring.

<u>Def</u> A <u>derived</u> scheme is a topological space X with a sheaf of topological comm. rings $\mathcal{O}_X$
s.t. $(X, \mathcal{O}_X) \underset{\text{locally}}{\simeq} (\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A})$
A topological commutative rings.

A topological commutative ring $\Rightarrow$
$\mathrm{Spec}\, A = \mathrm{Spec}\, \pi_0 A$ = Spec of underlying usual ring
$= \{ p \subset \pi_0 A \text{ prime ideals} \}$

Topology : Zariski opens $U_f = \{ p \not\ni f \}$ $f \in \pi_0 A$

Sheaf $\mathcal{O}_{\operatorname{Spec} A}(U_f) = \underline{\underline{A[f^{-1}]}}$ localization: makes perfect sense for topological comm. rings.

Improved Bezout's Theorem C, C' smooth varieties in projective space of any dimension.

[fundamental classes $\rightsquigarrow$ virtual fundamental class]

$[C] \cdot [C'] = [C \cap C']$ with no hypotheses.

...e.g could have $C = C'$.

Applications go both ways homotopy theory $\rightleftarrows$ algebraic geometry