J. Lurie - Derived Algebraic Geometry & Elliptic Cohomology 8/8/03

X space of homotopical interest ... e.g. BG or $K(G,n)$
... want to compute topological invariants....
e.g. $H^*(X)$ forms commutative ring (even case)
--- can think of as fns on algebraic variety
Spec $H^{2*}(X)$, use geometry .....

Even periodic cohomology theories:
E cohomology theory : $X \longmapsto E^*(X)$ graded ring
contravariant, Mayer-Vietoris, ...
<u>even periodic</u> :
$$E^n(*) = \begin{cases} 0 & n = 2k+1 \\ u^k \cdot E^0(*) & n = 2k \end{cases}$$
2-periodic, $u \in E^2(*)$ unit.

<u>Examples</u>: • E = K-theory. even-ness: ~~Stable~~ vector bundles on odd spheres are stably trivial.

• periodic version of cohomology:
$$HP^n(X) = \prod_k H^{n+2k}(X, \mathbb{Q})$$

Compute $E^0 := E(\mathbb{CP}^\infty)$ ... Atiyah-Hirzebruch s.s.
in terms of $E(*)$ & $H^*(\mathbb{CP}^\infty)$
- <u>no differentials</u> in ss. by even assumption

$\Rightarrow$ $E(\mathbb{CP}^\infty) \simeq E(*)[[t]]$ (noncanonical)
... analogous to $H^*(\mathbb{CP}^\infty, \mathbb{Z}) = \mathbb{Z}[t]$

$t = c_1$ of universal line bundle on $\mathbb{CP}^\infty$; $\mathcal{O}(1)$.

$\begin{matrix}\mathcal{L}\\ \downarrow \\ X\end{matrix}$ $\mathbb{C}$-line bundle on topological space $\Rightarrow$

$$\begin{matrix}\mathcal{L} & & \mathcal{O}(1)\\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & \mathbb{CP}^\infty\end{matrix}, \quad c_1(\mathcal{L}) := f^* t \in E(X)$$

Ordinary cohomology: $c_1(\mathcal{L}\otimes\mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$

Generally: look at universal case, $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$

$E(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq E(*)[[t_1, t_2]]$

$\Rightarrow c_1(\mathcal{L}\otimes\mathcal{L}') = f(c_1(\mathcal{L}), c_1(\mathcal{L}'))$ f power series in two variables

Commutativity of $\otimes$ $\Rightarrow$ $f(u,v) = f(v,u)$

associativity of $\otimes$ $\Rightarrow$ $f(u\, f(v\, w)) = f(f(u\, v)\, w)$

– formal group law over $E(*)$

K-theory: $c_1(\mathcal{L}) := 1 - [\mathcal{L}]$ (so trivial gets class 0)

$c_1(\mathcal{L}\otimes\mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}') - c_1(\mathcal{L})\, c_1(\mathcal{L}')$

formal multiplicative group

even periodic cohomology theories $\longrightarrow$ formal group over $E(*)$

cohomology theory $\overset{?}{\dashleftarrow}$ R ring, $\mathbb{G}$ f.g. $/R$

Landweber exact functor theorem: yes under mild hypotheses, & essentially unique answer

Example: $R = \mathbb{Z}$, $\mathbb{G}$ = multiplicative group $\longrightarrow$ complex K-theory

only 3 examples of 1-dim algebraic groups $/ k = \bar{k}$:

$\mathbb{G}_a \longleftrightarrow H^*$, $\mathbb{G}_m \longleftrightarrow K^*$, E elliptic curves $\longleftrightarrow$?

an <u>Elliptic cohomology theory</u> consists of

1. R commutative ring
2. elliptic curve E over R
3. A cohomology theory (even periodic) s.t. $A(*) \simeq R$
4. an isomorphism $\hat{E} \simeq \operatorname{Spf} A(\mathbb{CP}^\infty)$ as formal groups

Problems : 1. Too many examples : for every elliptic curve / ring R, ...

2. Construction of A from E, R not functorial enough ... can construct A functorially as a cohomology theory, but we want a representing spectrum Z,
$A(X) = [X, Z]$ (eg $H^* \Rightarrow$ E-M space, $K^* \Rightarrow \mathbb{Z} \times BU$)
--- we want Z not just up to homotopy equivalence but on the nose.

3. Want "good multiplicative structure" on A :
ring structure on $A \iff$ map $Z \wedge Z \longrightarrow Z$
commutative & associative up to homotopy : need coherent homotopies ! need A represented by an E_∞ ring spectrum.

Multiplicative groups : there are many such, one for any ring R - but can use just universal one, over $\mathbb{Z}$.
$\exists$ No universal elliptic curve over a ring, but there is one over a stack.

$\mathcal{M} = \mathcal{M}_{1,1}$ moduli stack (f étale)

$$\begin{array}{ccc} \operatorname{Spec} R \Rightarrow \text{Ell curve} & \Longrightarrow & A_f \text{ cohomology theory} \\ \downarrow f \text{ chart} & & \\ \mathcal{M} & \overset{?}{\dashrightarrow} & \{\text{spectra}\} \\ & & \downarrow \end{array}$$

$$\Rightarrow \left\{ \text{étale affines}/\mathcal{M} \right\} \longrightarrow \left\{ \text{cohomology theories} \right\}$$

Presheaf over $\mathcal{M}$, want global sections.
But can't sheafify or take global sections of cohomology theories, but only of spectra

... want to lift to spectra
Can solve this hard obstruction theory problem by making it harder ! try to lift to E_∞-ring spectra

Theorem (Goerss, Hopkins, Miller) $\exists(!)$ sheaf $\mathcal{O}^{der}$ of E_∞ ring spectra on $\mathcal{M}$ lifting $\{A_f\}$.

⇒ take global sections, get TMF spectrum.

Space of lifts is nonempty & connected --- so there is a unique object in a very weak sense, don't see $\mathcal{O}^{der}$ on the nose, & space of choices is not contractible. .....

E_∞ ring spectrum: homotopy theorist's commutative ring
-- vaguely, space A with structure of a comm ring
if we're ~~too~~ strict get back basically only ordinary cohomology.
Weaker: ring object in homotopy category! not strict or rich enough, don't have theory of rings, modules, algebras.
--- need homotopies of all levels.

$\pi_0 A$ underlying commutative ring

A derived scheme is a topological space X with a structure sheaf $\mathcal{O}_X$ of E_∞ ring spectra, s.t. locally $(X, \mathcal{O}_X) \simeq (\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A})$ $A \in E_\infty$-rings

A E_∞ ring spectrum ⇒ $\mathrm{Spec}\, A = \mathrm{Spec}\, \pi_0 A$ with Zariski topology
$\mathcal{O}_{\mathrm{Spec}\, A}(U_f) = A[f^{-1}]$

$U_f = \{p \not\ni f\}$ $f \in \pi_0 A$

Example: $(\mathcal{M}, \mathcal{O}^{der})$ derived Deligne-Mumford stack

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Elliptic Curve over R an E_∞ ring is a
1. $p: E \longrightarrow \mathrm{Spec}\, R$ map of derived schemes
2. p flat
3. $\bar{p}: \bar{E} \longrightarrow \mathrm{Spec}\, \pi_0 R$ (on underlying schemes) is an elliptic curves
4. E is a "very commutative" group object / R

An <u>orientation</u> of E is an equivalence between
$\hat{E}$ & $\mathrm{Spf}\ R^{\mathbb{CP}^\infty}$ ... maps $\mathbb{CP}^\infty \longrightarrow R$

<u>Theorem</u> $\mathrm{Hom}(\mathrm{Spec}\ A, (\mathcal{M}, \mathcal{O}^{der})) \simeq$
$\{$oriented elliptic curves $/A\}$

... in fact gives canonical characterization/construction of Hopkins-Miller $\mathcal{O}^{der}$, as representing a functor.