J. Lurie - Derived Algebraic Geometry & Elliptic Cohomology Toronto 10/04

R : an even periodic ring spectrum - $\pi_i(R) = 0$ i odd . Periodic: $\exists$ invertible element in $\pi_2 R$ $\beta \in \pi_2 R$ (Bott element in K-theory, e.g.)

$\Rightarrow R(\mathbb{C}P^\infty)$: Atiyah-Hirzebruch SS degenerates
$\cong$ noncanonically
$\pi_0 R[[x]]$

Even stronger statement: $\mathbb{C}P^\infty$ is comm group up to homotopy $\Rightarrow$ formal group law

let $A = \pi_0 R$

$\mathrm{Spf}\, R(\mathbb{C}P^\infty)$ is a formal group over A (commutative one-dimensional)

Reverse: start with comm ring A & formal group/A $\Rightarrow$ often construct even periodic ring spectrum.

Landweber exact functor thm - sufficient condition.

e.g. $\hat{\mathbb{G}}_m / \mathbb{Z}$ $\Rightarrow$ complex K-theory

Definition (Hopkins) An elliptic spectrum consists of

- even periodic ring spectrum R
- elliptic curve E over $\pi_0 R$
- an isomorphism $\mathrm{Spf}\, R(\mathbb{C}P^\infty) \simeq \hat{E}$ of formal groups.

Problems

1. Elliptic cohomology not functorial in elliptic curves - goes through Landweber, Brown representability, etc. not canonical construction on level of spectra
2. Want good (A_∞, E_∞ ...) multiplicative structures ... can't see from Brown rep. construction - maybe only see plain homotopy commutativity
3. Too many elliptic spectra! want universal choice

all addressed by Hopkins-Miller TMF construction

Algebraic Geometry: $\nexists$ universal elliptic curve over a ring, or a scheme, but have algebraic stack $\mathcal{M}$ moduli stack of elliptic curves $\mathcal{M}_{1,1}$

$\{\text{elliptic curves}/Y\} \longleftrightarrow \{\text{maps } Y \to \mathcal{M}\}$

Can cover $\mathcal{M}$ by "open pieces" which are affine schemes.

$U = \operatorname{Spec} A \xrightarrow{\eta} \mathcal{M}$ open — Landweber applies, so get elliptic spectrum $\boxed{\mathcal{O}^{der}(U)}$ corresponding to A & elliptic curve classified by η.

<u>So</u> to opens $U \longrightarrow$ cohomology theory: $\mathcal{O}^{der}$ is a presheaf on $\mathcal{M}$ with values in the <u>homotopy</u> category of spectra. $Ho(\text{Spectra})$

— Try to solve 1), 2) simultaneously: try to glue together multiplicative spectra, much more rigid, more tractable obstruction theory

<u>Theorem</u> (Goerss, Hopkins, Miller) $\mathcal{O}^{der}$ admits a lift — essentially unique — to a presheaf $\mathcal{O}^{der}$ with values in E_∞-ring spectra.

TMF $\sim$ global sections of $\mathcal{O}^{der}$.

Proof - Show space of all lifts $\mathcal{O}^{der}$ of $\mathcal{O}^{der}$ is nonempty & connected... so exists & unique up to some homotopy ... but this space is not simply connected. also can't expand this construction to other contexts, eg equivariant -

Derived Algebraic Geometry

Definition A scheme is a topological space X with a sheaf of rings $\mathcal{O}_X$ locally isomorphic to Spec A, A = commutative ring.

Replace commutative rings by functor homotopical version!

Def A derived scheme is a top space X with a sheaf of E_∞ ring spectra locally looks like Spec A, A = E_∞ ring spectrum

A E_∞ ring spectrum $\Rightarrow$ ~~A = HR~~
Spec A = Spec $\pi_0 A$ as a topological space

On basis of open nbhds U_f $(f \in \pi_0 A)$

$$\mathcal{O}_{\mathrm{Spec}\,\pi_0 A}(U_f) = (\pi_0 A)[f^{-1}]$$

$$\mathcal{O}_{\mathrm{Spec}\,A}(U_f) = A[f^{-1}] \qquad E_\infty \text{ ring spectrum}$$

X a derived scheme $\Rightarrow$ underlying ordinary scheme X^{ord}
e.g. $(\mathrm{Spec}\,A)^{ord} = \mathrm{Spec}\,\pi_0 A$.

So Goerss-Hopkins-Miller give a derived algebraic stack —

Keep Underlying "Space" of $\mathcal{M}$, give it $\mathcal{O}^{der}$ as new structure sheaf
$\Rightarrow \mathcal{M}^{der}$

What does $\mathcal{M}^{der}$ classify?

Suppose Y is a derived scheme.

Def An elliptic curve over Y is $p: E \longrightarrow Y$ morphism of derived schemes s.t.
1. p is flat
2. $E^{ord} \longrightarrow Y^{ord}$ is an elliptic curve/Y
3. E is a "very commutative" group (in strongest possible sense)

Group structure on derived elliptic curve <u>not</u> uniquely specified by choice of base point as in usual setting

$\mathcal{M}^{der}$ does <u>not</u> classify derived elliptic curve.....

<u>Def</u> An enriched elliptic spectrum consists of
- an E_∞ ring spectrum R
- an elliptic curve E over R
- an equivalence $\mathrm{Spf}(R^{\mathbb{CP}^\infty}) \xrightarrow{\sim} \hat{E}$

derived formal spectrum

<u>Theorem</u> For any E_∞ ring spectrum A, $\exists$ natural homotopy equivalence
$$\{\mathrm{Spec}\, A \longrightarrow \mathcal{M}^{der}\}^{\simeq} \longleftrightarrow \left\{\begin{array}{l}\text{enriched elliptic} \\ \text{spectra structures on } A\end{array}\right\}^{\simeq}$$

— Gives new construction of $\mathcal{M}^{der}$:
construct geometric object classifying this functor, easy, hard part is to identify the representing space has structure we want.

<u>K-theory analog</u>

R E_∞ ring spectrum.

Contemplate isomorphism $\mathrm{Spf}\, R^{\mathbb{CP}^\infty} \xrightarrow{\sim} \hat{\mathbb{G}}_m$

Step 1: Have a map $\mathrm{Spf}\, R^{\mathbb{CP}^\infty} \longrightarrow \hat{\mathbb{G}}_m$

$\Longleftrightarrow$ $\mathrm{Spf}\, R^{\mathbb{CP}^\infty} \longrightarrow \mathbb{G}_m$

Spf takes products into maps!

this is a $\mathbb{CP}^\infty$-parametrized family of maps from $\mathrm{Spec}\, R \longrightarrow \mathbb{G}_m$

$\Longleftrightarrow$ element in $\mathbb{G}_m(R)$

So we're classifying (E_∞) maps $\mathbb{CP}^\infty \longrightarrow GL_1(R)$
$\Longleftrightarrow$ $\Sigma^\infty \mathbb{CP}^\infty \longrightarrow R$

<u>Step 2</u> When is this an isomorphism?
$\mathbb{CP}^1 = S^2 \subset \mathbb{CP}^\infty \longrightarrow \Sigma^\infty \mathbb{CP}^\infty$ canonical elt of $\beta \in \pi_2(\Sigma^\infty \mathbb{CP}^\infty)$
... need β invertible

Theorem (Snaith): $\Sigma^\infty \mathbb{CP}^\infty[\beta^{-1}] \simeq K$.

interpretation: K-theory classifies isomorphisms $\hat{\mathbb{G}}_m$ & Spf $R^{\mathbb{CP}^\infty}$

Theorem pins down TMF canonically, up to contractible space of choices.

e.g. string orientation of TMF comes canonically here, as do its equivariant versions.
Equivariant elliptic cohomology:
have universal elliptic curve / M^{der},
encoding S^1-equivariant elliptic cohomology.
Moduli over it give G-equivariant elliptic cohomology for G compact.