MSRI
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I. Mirković - Perverse Sheaves on a Loop Grassmannian
(Drinfeld/Lusztig/Ginzburg/Mirković-Vilonen)

G algebraic group, $a \subset C$ finite subscheme of curve

$\Rightarrow$ loop grassmannian $\mathcal{G}_a = H^1_a(C, G)$ local cohomology at subscheme a
$= G$-torsors $P \to C$ + trivialization off a

$a = \text{point} \in C$: $\mathcal{G}_a = \left(\text{torsors + triv on } \hat{a} \text{ (formal nbhd)} + \text{triv on } C - a \right) / G(\hat{a})$ sections on formal nbhd

$= G(\hat{a} - a) / G(\hat{a})$

loop group: punctured formal nbhd — positive loops

z local parameter at a $\Rightarrow$ $\mathcal{O} = \mathbb{C}[[z]]$, $\mathcal{K} = \mathbb{C}((z))$

$\Rightarrow \mathcal{G}_a = G(\mathcal{K}) / G(\mathcal{O})$.

Relation with Langlands: $G(\mathcal{O}) \backslash \mathcal{G}_a$ orbits = data for modifying G-torsors on C at point a

To modify perverse sheaves on moduli of G-torsors
$\Rightarrow$ consider $G(\mathcal{O})$-equivariant perverse sheaves

$\mathcal{P}_{G(\mathcal{O})}(\mathcal{G}_a) = \text{Rep } G^\vee$

$P[\mathcal{B}un_G(C)]$ $\to$ perverse sheaves on moduli of G-torsors

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Basic Result [assume $a = 0 \in \mathbb{A}^1$, C = formal nbhd of $0 \in \mathbb{A}^1$]

Perverse sheaves (equivariant) $\mathcal{P}_{G(\mathcal{O})}(\mathcal{G}, k)$ coefficients in k-modules

[$\mathcal{K} = \mathbb{F}((t))$ $\mathbb{F}$ could be $\mathbb{C}$ or $\mathbb{F}_q$
— when $\mathbb{F} = \mathbb{C}$ can take k any commutative ring, nötherian of finite dim
— when $\mathbb{F} = \mathbb{F}_q$ take $k = \overline{\mathbb{Q}}_\ell$]

$\mathcal{P}_{G(\mathcal{O})}(\mathcal{G}, k) \xrightarrow{\sim}$ Algebraic reps $\text{Rep}(G^\vee_k) \to$ split form

$$\begin{tikzcd} \mathcal{P}(\mathcal{G}, k) \arrow[rr, "\sim"] \arrow[dr, "H^\bullet(\mathcal{G}, -)"'] & & \mathrm{Rep}(G_k^\vee) \arrow[dl, "\mathrm{Forget}"] \\ & \mathrm{Mod}(k) & \end{tikzcd}$$

i.e. total cohomology will have action of $G_k^\vee$.

$\otimes$ on reps $\longleftrightarrow$ $*$ on perverse sheaves:
imitate convolution product $\mathbb{C}_{B \times B}[A]$
B - bi-invariant Fs on A
$\cup$ $G(O)$ $\quad$ $\cup$ $G(K)$
- disadvantage: not obviously commutative convolution.

Fusion approach: C global curve (eg $\mathbb{A}^1$)
look at finite Hilbert scheme $C^{[n]}$
& deform $\mathcal{G}r$ to

$$\begin{array}{ccc} \mathcal{G}_{C^{[n]}} & \supset & \mathcal{G}_a \\ \downarrow & & \downarrow \\ C^{[n]} & \supset & a \end{array}$$

$C^{[n]} = C^{(n)} \longleftarrow C^n$: pull back to nth power of curve, get version $\mathcal{G}_{C^n}$ of $\mathcal{G}$ over C^n.

Theorem a. As ind-scheme over C^n, $\mathcal{G}_{C^n}$ is flat.

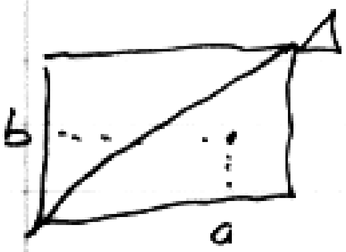
b. fibers (case $n=2$) $\mathcal{G}_{a,b} = \begin{cases} \mathcal{G}_a \times \mathcal{G}_b & a \neq b \\ \mathcal{G}_a & a = b \end{cases}$

- huge jump - but on finite dimensional pieces products converge to something of right size over diagonal.

- this is just locality of local cohomology!

$$H^1_{D^1 \amalg D^2}(C, G) = H^1_{D^1}(C, G) \times H^1_{D^2}(C, G)$$

(or ind-scheme version thereof...)

C^2: 

$\mathcal{G}_a \times \mathcal{G}_b$

<u>Convolution</u>:

On $\mathcal{G}_a \times \mathcal{G}_b$ put exterior product of sheaves $A \boxtimes B$

as $a \to b$ get limit $A * B$ on $\mathcal{G}_a$

limit := <u>nearby cycles</u> at Δ.

- $*$ is now manifestly commutative!

$$A, B \in P_{G(O)}(\mathcal{G}) \longrightarrow A * B \in P_{G(O)}(\mathcal{G})$$

Get construction of $\mathcal{O}(G_k^\vee)$ & of $U(\hat{n}_k)$ geometrically here

Algebraically: Weyl modules $W_\lambda \twoheadleftarrow V_\lambda$ Verma

Geometrically: coWeyl $\check{W}_\lambda = \Gamma(\check{\mathcal{B}}, \mathcal{O}_\lambda)$ line bundle on affine flags

(k field) $\mathrm{Irr}(G_k^\vee) \longleftrightarrow$ orbits $G(O)\backslash\mathcal{G}$

$$W_\lambda \longleftrightarrow \text{orbit } \mathcal{G}_\lambda \subset \mathcal{G}$$

To each orbit have 3 kinds of perverse sheaves

$I_!(\mathcal{G}_\lambda, k)$: take shifted constant sheaf $k_{\mathcal{G}_\lambda}[\dim]$

$H^0_{Perv}(j_! \, k_{\mathcal{G}_\lambda}[\dim])$

$I_* = H^0_{Perv}(j_* \, k_{\mathcal{G}_\lambda}[\dim]$

$I_! \longrightarrow I_*$ and image is denoted by $I_{!*}$

$I_! \twoheadrightarrow I_{!*} \hookrightarrow I_*$

Total cohomology of these $H^0(\mathcal{G}, I)$:

$I_! \longleftrightarrow \check{W}_\lambda$ coWeyl (k coeffs)

$I_* \longleftrightarrow W_\lambda$ Weyl

$I_{!*} \longleftrightarrow L_\lambda$ irred module

(works for any k - really over $\mathbb{Z}$!)

$\mathcal{G} = G(K)/G(O)$ partial flag variety $\quad B = TN$

$G(K) \supset G \supset T$ torus, fix two Borels $B_- = TN_-$

Cartan fixed points $\mathcal{G}^T \longleftrightarrow X_*(T)$ cocharacters

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\cup\!\!\!\shortmid$ $\quad\quad\quad\quad\quad\quad\quad$ $\uparrow$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ [elt of $T(K)$] $\longleftrightarrow$ loop into T

$\lambda \in X_*(T) \longleftrightarrow L_\lambda \in \mathcal{G}^T$ fixed point.

Three kinds of Borel: Iwahori: $I = \left(G(\mathcal{O}) \underset{\text{eval at } 0}{\twoheadrightarrow} G\right)^{-1}(B)$

$I^- = \left(G(\mathbb{C}[z^{-1}]) \underset{\text{eval at } \infty}{\longrightarrow} G\right)^{-1}(B)$

$J = T(\mathcal{O}) \cdot N(K)$

For each of these, orbits on $\mathcal{G}$ indexed by fixed pts $\mathcal{G}^T$ i.e. cocharacters

For I orbits fin dim $\quad I^-$ fin codim

J semi-infinite: ∞ dim $\underline{\&}$ codim.

$G(\mathcal{O}) \supset I$ slightly bigger $\rightsquigarrow G(\mathcal{O})$ orbits labelled by $X_*(T)/W \ni \lambda \longmapsto \mathcal{G}_\lambda = G(\mathcal{O}) \cdot L_\lambda$.

<u>Examples of orbits</u>: 0. Each orbit is a vector bundle over the $\mathbb{G}_m$-fixed points $\quad \mathcal{G}_\lambda \longrightarrow \mathcal{G}_\lambda^{\mathbb{G}_m}$

$\mathbb{G}_m$ = ~~rotating~~ loop

$\ni s \quad (s\circ\lambda)(z) = \lambda(s^{-1}z)$

$\mathcal{G}_\lambda^{\mathbb{G}_m}$ is a partial flag variety for finite G

$\rightarrow$ $\mathcal{G}$ obtained by gluing these.

1. nilpotent cone $\mathcal{N} \hookrightarrow \mathcal{G}$: x nilpotent $\Rightarrow$

$x \longmapsto \boxed{e^{z^{-1}x} \cdot L_0} \in \mathcal{G}$

GL_n case: closure of orbit for first fund weight

$\mathcal{G}_{\varpi_1}$ = compactification of nilpotent cone in $n\times n$ matrices $\mathcal{N} \subset M_n$

So rel positions of G orbits on $\mathcal{N}$ and on Gr correspond

2. Open part of $\mathcal{G}(Gl_n)$ is normal slice in nilpotent operators on $\mathbb{C}^n$, at operator Z.

$\Rightarrow$ orbits in $\mathcal{G}(Gl_n)$ $\longleftrightarrow$ geometry of all nilpotent cones put together.

3. $G = SL_2$ $\quad G^\vee = PSL_2$ $\quad \to$ orbits $\mathcal{G}_0, \mathcal{G}_{\alpha^\vee}, \mathcal{G}_{2\alpha^\vee}$

$\overline{\mathcal{G}_{n\alpha^\vee}}$ looks like a projective space: it's union of $\mathbb{A}^n \cup \dots \cup \mathbb{A}^0$ but not smooth:
have action of $Sl_2(\mathbb{Z})$ [illegible] on cohomology, unlike $H^\cdot(\mathbb{P}^{n-1}, \mathbb{Z})$

eg $\overline{\mathcal{G}_{\alpha^\vee}}$ = cotangent bundle to $\mathbb{P}^1$ union one point
$\sim \mathbb{P}^1 \cup \mathcal{N}_2$ nilpotent 2×2 matrices

$\mathcal{C}_0$
$\mathcal{C}_{\alpha^\vee}$

$\mathbb{P}^1$

singularity is $\mathbb{C}^2/\pm 1$ problem with 2-torsion $\Rightarrow$ over $\mathbb{F}_2$ the dimension of $L_{\alpha^\vee}(\mathbb{F}_2) = 2$ rather than 3 as expected.

Now $G = PSl_2$, $G^\vee = Sl_2$: study tensor product of 2-dim reps $\mathbb{C}^2 \otimes \mathbb{C}^2$
$\mathbb{C}^2 = H^\cdot(\mathbb{P}^1, \mathbb{C})$ $\quad$ study degeneration $\mathcal{G}_{a,b} \longrightarrow \mathcal{G}_a$

$\mathbb{P}^1 \times \mathbb{P}^1$ degenerating to singular quadric in $\mathbb{P}^3$. (nilcone)

$$\mathcal{G}_{\mathbb{P}^1\times\mathbb{P}^1} = \mathcal{G}_{\rho^\vee} \times \mathcal{G}_{\rho^\vee} \rightsquigarrow \overline{\mathcal{G}_{2\rho^\vee}}$$

but can degenerate differently into $\mathbb{P}^1$ bundle over $\mathbb{P}^1$: Springer resolution of nilcone
... get convolution in usual picture for functions...
identity of these two descriptions $\longleftrightarrow$ compatibility of constructions of convolution

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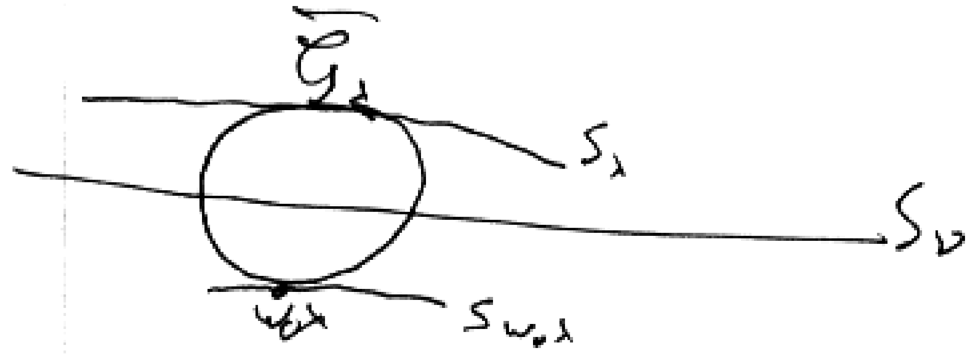
Basic technique: $\quad$ comparison of 2 types of Schubert cycles

Lemma $\quad \overline{\mathcal{G}_\lambda} \cap \overline{S_\nu}$ is of pure dimension $ht(\lambda+\nu)$ (λ dominant)

$\overline{\mathcal{G}_\lambda}$ = closure of $G(\mathcal{O})$ orbit

$\overline{S_\nu}$ = $T(\mathcal{O}) N(\mathcal{K}) \cdot L_\nu$ $\quad \frac{\infty}{2}$-orbit (of "Borel" $\mathcal{I}$)

ht = contract with ρ



intersection = $\cup$ of irred components of same dimension.

$\nu = \lambda$: intersection is open in fact Iwahori orbit $I \cdot \lambda \subset \bar{\mathcal{G}}_\lambda$

Opposite case $\nu = w_0 \lambda$: intersection is just one point $w_0 \cdot \lambda$

For general intersections write chain $\nu_0 \cdots \nu_i \cdots \nu_n$, with $\nu_0 = \lambda$, $\nu_i = \nu$, $\nu_n = w_0\lambda$

all S_ν have boundaries given by one equation
$\Rightarrow$ dim of irred components drop by one (or stay same) at any stage.

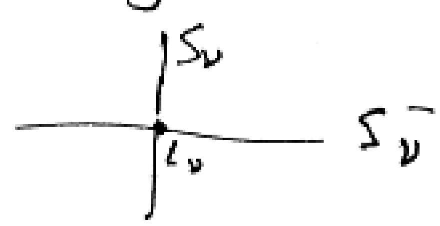
Consequences 1. $H^\bullet_c(S_\nu, \mathcal{A})$ compactly supported
$\mathcal{A} \in P_{G(\mathcal{O})}(\mathcal{G})$
is in only degree $2ht\,\nu$.

--- restrict $\mathcal{A}$ to $\mathcal{G}_\lambda$ for different λ,
use perversity estimates : degrees $\leq -2ht\,\lambda$
Take $H^\bullet_c(S_\nu \cap \mathcal{G}_\lambda, \mathcal{A}) \Rightarrow$ degrees $\leq -2ht\,\lambda + 2ht(\nu+\lambda)$
perturb by dim of intersection $= 2ht\,\nu$

$\Rightarrow$ easy estimate on one side.
To get other side : $H^\bullet_c(S_\nu, \mathcal{A}) = H^\bullet_{S_{\bar\nu}}(\mathcal{G}, \mathcal{A})$

– local cohomology for negative orbit : $T(\mathcal{O})\, N_-(\mathcal{K}) \cdot L_\nu = S_{\bar\nu}$, with $T(\mathcal{O})\, N_-(\mathcal{K}) = J_-$, $J_- \cdot L_\nu$

– dual stratification



$\Rightarrow$ restriction by $*$ to orbit, then ! to pt $\longleftrightarrow$
" " ! to transversal orbit, then $*$ to pt !

Consequence : $H^\bullet(\mathcal{G}, \mathcal{A}) = \bigoplus_\nu H^\bullet_c(S_\nu, \mathcal{A})$
$\nu \in X_*(T)$ grading $\rightarrow$ so we have action of dual Cartan

⇒ <u>Canonical basis</u> of representations:

$$H^*_c(S_\nu, I_!(\mathcal{G}_\lambda, k)) = k[\text{Irred Components}(\mathcal{G}_\lambda \cap S_\nu)]$$
$$\| $$
$$W_\lambda(\nu) \quad \nu\text{-weight space} \Rightarrow \text{basis!}$$

Conjecture these irred components determined by fixed points of torus. J. Anderson: think of these f. points as coordinates & connect the dots

eg sl_3

six pictures for 6 basis elts:

→ read off branching rules etc

Orbits <u>even</u> <u>dimensional</u>: gives hope that IC might have basis of alg cycles!