I Mirković - Lie algebras in positive characteristic: geometry & Langlands duality 8/12/08

Global (1. Unramified global geometric Conjecture (Beilinson, Drinfeld)

local ingredients we know something about (2. Loop Grassmann of G & reps of G^L (Drinfeld, Ginzburg, Lusztig, Mirković, Vilonen, Finkelberg, Bezrukavnikov)

3. Affine flag variety of G & geometry of G^L (Bezr. w/ Arkhipov, Ginzburg)

applications 4. Lie algebras for $p > 0$, exotic coherent sheaves on Springer fiber (Bez - M. - Rumynin)

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1. Class field theory : F # field, $\mathbb{F}_q(X)$...
$(Gal_F)^{ab}$ described

Langlands: huge extension, describing full Gal_F
... reductive groups $G \longleftrightarrow G^L$

Geometric Langlands: forget # fields, concentrate on geometric case. Pass to more sophisticated objects
Automorphic function (Hecke eigenfunction) ......>
? comes from perverse sheaf with analogous properties.
... parallel to studying directly automorphic functions.

Beilinson formulation: $k = \bar{k}$ arbitrary, X/k projective smooth connected curve

D^b Perverse $\mathrm{Bun}_G(X) \xleftarrow{F} \mathrm{LocSys}_{G^L}(X) \leftarrow D^b(\mathrm{Coherent})$

Fourier (Topology o Alg. geom) $\Longrightarrow$ Alg geom (topology)

Understanding: very very beginning.
Traditional content: Galois rep $\Rightarrow$ loc sys σ on X
$\longmapsto F(\sigma)$ Hecke eigensheaf, given by Fourier transform applied to skyscraper $\mathcal{O}_\sigma$

V vector bundle : $V \underset{\mathcal{O}_{LS}}{\otimes} \mathcal{O}_\sigma = V_\sigma \underset{k}{\otimes} \mathcal{O}_\sigma$ eigenproperty!

Canonical vector bundles on $\mathcal{LS}$ moduli of local systems : associated to pair $V \in \mathrm{Rep}\, G^\vee$ & $a \in X$

$$(\mathcal{V}_{V,a})|_\sigma = (V)_{\sigma_a}$$

σ $G^\vee$-local system

$(V)_{\sigma_a}$ twist of V by σ at point a

Automorphic side: $\mathcal{F}(\mathcal{O}_\sigma)$ perverse sheaf on Bun_G

$\otimes$ $\dashrightarrow$ convolution

So want $\mathcal{F}(V,a) * \mathcal{F}(\mathcal{O}_\sigma) = (\mathcal{V}_{V,a})_\sigma \otimes \mathcal{F}(\mathcal{O}_\sigma)$

some kind of operator

some kind of convolution

On Bun_G side: have modifications of G-torsors / Hecke correspondences:

$a \in X$ $\quad \mathcal{H}_a = \{(P, Q, P \xrightarrow{\sim} Q \text{ off } a)\}$

$P \in \mathrm{Bun}_G \swarrow \qquad \searrow \mathrm{Bun}_G \ni Q$

Types of modifications : look at case P = trivial.

$\Rightarrow \{Q$ G-bundle + trivialization off $a\} =: \mathcal{G}_a$

loop Grassmannian at a.

$\mathcal{G}_a$: $\quad \hat{a}$ = formal nbhd of $a \in X$, $\quad \overset{\circ}{\hat{a}} = \hat{a} \setminus a$

$\quad = G^{\overset{\circ}{\hat{a}}} / G^{\hat{a}}$ $\qquad$ loop group / arc group

vs transition functions on $\overset{\circ}{\hat{a}}$ mod reparam.

Types of modifications $\longleftrightarrow G^{\hat{a}} \backslash \mathcal{G}_a \longleftrightarrow \mathrm{Irr}\, G^\vee$

Convolution operators: $V \in \mathrm{Irr}\ G^\vee \longmapsto$
topological object: irreducible perverse sheaf $\mathcal{F}(V)$
on $\mathcal{G}_a$, $\mathcal{F}(V) = IC(\text{"}\overline{G^{\hat{a}} \cdot V}\text{"})$ orbit closure in Grassmann

$$[\mathcal{F}(V) * \mathcal{E}](Q) = \int_{\substack{m \in \mathcal{G}_a \\ \text{modifications}}} \mathcal{E}[m(Q)] \cdot \mathcal{F}(V)_m$$

weight values of $\mathcal{E}$ at modifications by IC sheaf $\mathcal{F}(V)$.

Note: $\mathcal{L}\mathcal{S}$ really stacks, so points σ really determines <u>category</u> of coherent sheaves sitting on σ
$\Rightarrow$ <u>category</u> of Hecke sheaves assigned to σ inside $\mathcal{P}(\mathrm{Bun}_G)$

2. Loop Grassmannians have multiple personality:
- modifications
- simple cohomological objects
- group theory

Cohomological: $\mathcal{G}_a = H^1_a(X, G)$ local cohomology

<u>Theorem</u> $\mathcal{P}_{G^{\hat{a}}}(\mathcal{G}_{a,k}) \simeq \mathrm{Rep}\ G^\vee_k$

perverse sheaves with any coefficient ring k (e.g. $\mathbb{Z}$)
Drinfeld's idea for convolving perverse sheaves:

$$H^1_{\{a,b\}}(X, G) = H^1_a(X, G) \times H^1_b(X, G) \quad \text{for } a \neq b$$

$$a, b \to c \Rightarrow H^1_c(X, G) \Rightarrow \mathcal{G}_a \times \mathcal{G}_b \rightsquigarrow \mathcal{G}_c$$

$$\mathcal{A} * \mathcal{B} = \lim (\mathcal{A} \boxtimes \mathcal{B})$$

$$\Rightarrow \mathcal{P}_{G^{\hat{a}}} = \mathrm{Rep}\,(\mathrm{Aut}\,[H^\bullet(\mathcal{G}_a, -)])$$

- Calculate this $= \check{G}$. $\Rightarrow$ "only" construction of $\check{G}$.

Physics: recovering $G^\vee$ from collisions of G-particles
... use locality of QFT on curve
$\mathcal{G}_a$ is a geometric conformal field theory

⇒ <u>Canonical basis</u> : $V = IH^*(\overline{G^{\hat{a}} \cdot V})$ "Schubert cell" cohomology

also have $\infty/2$ Schubert cells ⇒ canonical basis for weight spaces of V from intersections of finite & $\infty/2$ orbits.

$V = H^0(G^\vee/B^\vee/\mathcal{L}_\lambda)$; canonical basis : maybe $G^\vee$ defined over $\mathbb{Z}_+$ or more set theoretically?!

3. Affine flags:

$$\begin{tikzcd} I = \text{Iwahori subgroup} \arrow[r] \arrow[d] & G^{\hat{a}} \arrow[d, "\text{eval at } a"] \\ B \arrow[r] & G \end{tikzcd}$$

$\mathcal{P}_I(G^{\tilde{a}}/I)$

$\mathcal{P}_{G^{\hat{a}}}(G^{\tilde{a}}/G^{\hat{a}}) = \mathrm{Coh}_{G^\vee}(pt) = \mathrm{Rep}\, G^\vee$

$$D^b(\mathcal{P}_I(G^{\tilde{a}}/I)) \simeq D^b(\mathrm{Coh}_{G^\vee}(St_{G^\vee})) \quad \text{Steinberg}$$

$$\begin{tikzcd} T^*(\mathcal{B}^\vee = G^\vee/B^\vee) \arrow[r, phantom, "\subset"] \arrow[d] & \tilde{\mathfrak{g}}^\vee \arrow[d] \\ \mathcal{N}^\vee \arrow[r, phantom, "\subset"] & \mathfrak{g}^\vee \end{tikzcd}$$

$St = T^*\mathcal{B}^\vee \times_{\mathcal{N}^\vee} T^*\mathcal{B}^\vee$

4. $G^\vee$ semisimple / $k = \bar{k}$ char $p > 0$, $\mathfrak{g} = \mathrm{Lie}\, G$

$\mathrm{Rep}\, \mathfrak{g}^\vee \simeq \mathrm{Coh}\,(?)$
‖
$\mathrm{Rep}\, U\mathfrak{g}^\vee \supset \mathrm{Rep}^\lambda_\chi(U\mathfrak{g}^\vee)$ fix action of center

$D^b(\mathrm{Rep}^\lambda_\chi U\mathfrak{g}^\vee) \simeq D^b\,\mathrm{Coh}(\mathcal{B}^\vee_{\chi,\lambda}) \to$ Springer fibers

$$D^b \operatorname{Coh} \mathcal{B}^\vee_{\lambda, x} \simeq D^b [P_I(Fl_G)]^\vee_x$$

Lusztig combinatorics: translate from rep theory ($U_q \mathfrak{g}^\vee$-mod)
to algebraic geometry ($\operatorname{Coh} \mathcal{B}^\vee$)
to topology ($P_I(Fl_G)$), use Hodge theory ...

- Rep theory of semisimple Lie algebras = some resolution in algebraic geometry.
- Critical quantization

(trivial rep theory) 1. Crystalline differentials on X/k $[p \geq 0]$

$$\mathcal{D}_X = \langle \mathcal{O}_X, T_X, [\partial, f] = \partial(f) \rangle$$

$$Z(\mathcal{D}_X) = \begin{cases} \text{const functions} & p=0 \\ \mathcal{O}_{T^* X^{(1)}} \quad \text{Frobenius twist} & p>0 \end{cases}$$

$[\mathcal{O}_{Y^{(1)}} = (\mathcal{O}_Y)^p]$

- $\mathcal{D}_X$ is Azumaya over $T^* X^{(1)}$
- Sometimes splits: e.g. on conormals to smooth subvarieties

(nontrivial rep theory) 2. $U\mathfrak{g} \rightsquigarrow S\mathfrak{g} = \mathcal{O}(\mathfrak{g}^*)$: but $\cancel{\mathcal{O}_{\mathfrak{g}^*}}$

$\mathfrak{g}^*$ very singular as Poisson variety, much harder than $T^*X \rightsquigarrow \mathcal{D}_X$.

To attack this just need Poisson resolution
- smooth Poisson: all leaves closed of same size

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \supset & T^*\mathcal{B} \\ \downarrow & & \downarrow \\ \mathfrak{g}^* & \supset & \mathcal{N} \end{array}$$

$\tilde{\mathfrak{g}} \to \mathfrak{g}$ Finite-to-one generically, but get resolution on each leaf T^*_λ

Quantize $\mathcal{O}(\tilde{\mathfrak{g}}) \Rightarrow \mathcal{D}_{\mathcal{B}}$ deformation

$U\mathfrak{g} \to \mathcal{D}_{\mathcal{B}}$

$$D(\mathcal{D}^{\lambda}_{\mathcal{B}}\text{-mod}) \xrightarrow[\simeq]{\Gamma} D(U\mathfrak{g}^{\lambda}\text{-mod})$$

1. λ regular 2. $p > h_{\mathfrak{g}}$

$\cancel{D^b(\mathcal{D}_{\mathcal{B}}}$

$$D^b[\mathrm{Mod}_{\mathcal{B}_\chi}(\mathcal{D}^{\lambda}_{\mathcal{B}})] \simeq D^b(\mathrm{Mod}^{\lambda}_{\chi}\, U\mathfrak{g})$$

$$\|$$

$$D^b(\mathrm{Coh}\,\mathcal{B}_\chi]$$