Note Title 8/30/2007

Mitya Boyarchenko - Characters of unipotent groups over finite fields

1. General remarks

Def An algebraic group over a field k is a smooth group scheme of finite type $/k$

If G is an algebraic group over $\mathbb{F}_q$ then the set $G(\mathbb{F}_q)$ is a finite group

Goal: study representations$_{\mathbb{C}}$ of $G(\mathbb{F}_q)$ in terms of the geometry of G

There are at least two approaches:

1. Try to realize irreducible reps of $G(\mathbb{F}_q)$ in cohomology of algebraic varieties $H^i(X, \overline{\mathbb{Q}_\ell})$ ($X/\mathbb{F}_q$ with G action)
$\rightsquigarrow$ Deligne-Lusztig theory for G reductive

2. Relate irreducible characters of $G(\mathbb{F}_q)$ over $\overline{\mathbb{Q}_\ell}$ to the trace functions

associated to some $\overline{\mathbb{Q}}_\ell$-sheaves on G
(Lusztig's character sheaves, for G reductive)

Reminder about sheaves to functions correspondence:

X = scheme of finite type $/\mathbb{F}_q$
ℓ prime $\neq$ char $(\mathbb{F}_q)$
$\Rightarrow$ well behaved notion of a constructible $\overline{\mathbb{Q}}_\ell$ sheaf on X

If M is such a sheaf $\rightsquigarrow$ function
$$t_M : X(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$$
$$t_M(x) = \mathrm{tr}\,(\varphi^{-1}; x^* M)$$
$x : \operatorname{Spec} \mathbb{F}_q \to X$, $x^*(M)$ is a fin dim $\overline{\mathbb{Q}}_\ell$-rep of $\mathrm{Gal}\,\overline{\mathbb{F}}_q/\mathbb{F}_q \ni \varphi$ = Frobenius
$\varphi(a) = a^q$

Example! Artin-Schreier local system :

$$0 \to (\mathbb{F}_q, +) \to \mathbb{G}_a \to \mathbb{G}_a \to 0$$
$$x \longmapsto x^q - x$$

over $\mathbb{F}_q$

This map is an étale cover of $\mathbb{G}_a$
(in fact Galois for $(\mathbb{F}_q, +)$)

So given a character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$
the Artin-Schreier cover gives a rank 1 local system defined by ψ,
& its trace function $\boxed{t_{\mathcal{L}_\psi} = \psi}$

Generalization: replace $\mathbb{G}_a$ by any commutative connected group over $\mathbb{F}_q$
Artin-Schreier $\rightsquigarrow$ Lang isogeny

$$0 \to G(\mathbb{F}_q) \to G \xrightarrow{L} G \to 0$$
$$g \longmapsto Fr(g) - g$$

So any character of $G(\mathbb{F}_q)$ (G commutative) can be promoted to a sheaf on G !

We want to study irreducible characters of $G(\mathbb{F}_q)$, G a connected unipotent group over $\mathbb{F}_q$.

<u>Recall</u>: If G is a unipotent group $/\mathbb{R}$, ($\leftrightarrow$ connected & simply connected nilpotent Lie group)

we have Kirillov's orbit method, describing $\widehat{G}$ in terms of coadjoint action $G \circlearrowright \mathfrak{g}^*$.

In particular every unitary irrep of G can be realized in L^2 sections of a G-equivariant unitary line bundle on some homogeneous space X for G.

<u>Very naive analog</u>

G = connected unipotent group $/\mathbb{F}_q$

Can hope that any irrep of $G(\mathbb{F}_q)$ can be realized in the space of functions $\{X(\mathbb{F}_q) \to \mathbb{C}\}$ where $X/\mathbb{F}_q$ a variety with a transitive G-action.

There is a class for which this can always be done ...

If this could be done $\Rightarrow$ the dim of any irrep of $G(\mathbb{F}_q)$ would have to be a power of q. (since $X \simeq \mathbb{A}^d$ for some d)

This is not always the case!

<u>First counterexample</u> (Lusztig, 2003)

maximal unipotent of symplectic group $/\mathbb{F}_2$

If $G =$ max unipotent of $Sp(4)$ over $\mathbb{F}_q$ where q is an even power of 2

$\rightarrow$ have irreps of dimension $q^{1/2}$.

<u>A different class of counterexamples</u>:

"fake Heisenberg groups" over $\mathbb{F}_q$

$$1 \longrightarrow \mathbb{G}_a \longrightarrow G \longrightarrow \mathbb{G}_a \longrightarrow 1$$

nontrivial central extension of $\mathbb{G}_a$!

any noncommutative G of this form is called a fake Heisenberg group. (works in any characteristic)

2. Unipotent groups with connected centralizers

<u>Def</u> An algebraic group G/k has connected centralizers if the centralizer of any geometric point $g \in G(\bar{k})$ in $G \otimes_k \bar{k}$ is connected.

<u>Main Theorem</u> Let G be a unipotent group over $\mathbb{F}_q$ with connected centralizers. For any irrep ρ of $G(\mathbb{F}_q)$ there exist a connected closed subgroup $H \subset G$ and a one-dim representation χ of $H(\mathbb{F}_q)$ s.t.

$\rho = \mathrm{Ind}_{H(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\chi)$ ($\Rightarrow$ $\dim \rho$ is a power of q, in fact $= q^{\dim G - \dim H}$)

Historical comments

Let A be a fin-dim associative algebra over $\mathbb{F}_q$. Define

$$G(A) = \{ 1+x : x \in \text{nil radical of } A \}$$

This is the set of $\mathbb{F}_q$-pts of a unipotent group $/\mathbb{F}_q$ with connected centralizers.

(all centralizers are linear subspaces of A).

Key example: $UL_n = \left\{ \begin{pmatrix} 1 & * \\ & \ddots \\ & & 1 \end{pmatrix} \right\}$ unipotent linear group

comes from $A =$ upper triangular matrices.

J. Thompson conjecture (1965 ?) that every irrep of UL_n has dimension a power of q

1973 Gutkin's incorrect proof of:

Any irrep of an algebra group is induced from a 1-dim rep of a sub-algebra group.

(ie $G(B) \subset G(A)$ for $B \subset A$)

1995 Isaacs proved that the dim of every irrep of $G(A)$ is a power of q

2004 Halasi proved Gutkin's "conjecture"

General picture

In representation theory of finite groups there are two important facts:

1. Γ finite group, $\mathrm{Fun}(\Gamma)$ = functions on $\Gamma \to \mathbb{C}$ under convolution

   The center of this is $\mathrm{Fun}(\Gamma)^{\Gamma}$ = conjugation invariant functions

   There's a natural bijection

$$\begin{array}{ccc} \text{irred characters of } \Gamma & \longleftrightarrow & \text{minimal idempotents in } (\mathrm{Fun}\,\Gamma)^{\Gamma} \\ \chi & \longmapsto & \frac{\chi(1)}{|\Gamma|} \cdot \chi \end{array}$$

2. If Γ is a finite nilpotent group
$\Rightarrow$ every irrep of $\Gamma / \mathbb{C}$ is
induced from a 1-d rep of a subgroup

Geometrizing 1 leads to a definition of
L-packets of irreducible characters of $G(\mathbb{F}_q)$

Geometrizing 2 leads to a concrete description
of L-packets

How do we geometrize 2 ?

First geometrize the notion of a 1-dim
representation: H any algebraic group / $\mathbb{F}_q$
we define a multiplicative local system on H
to be a $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}$ on H
s.t. $\mu^*(\mathcal{L}) \cong \mathcal{L} \otimes \mathcal{L}$, $\mu : H \times H \to H$
multiplication

Corresponding functions are homomorphisms $G(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^*$

If H is commutative & connected then any

homomorphism arises this way
- but usually false for eg fake Heisenberg groups.

<u>Main Theorem 1'</u> G = unipotent group / $\mathbb{F}_q$ with connected centralizers $\Rightarrow$ every irrep of $G(\mathbb{F}_q)$ is of the form $\mathrm{Ind}_{H(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(t_{\mathcal{L}})$ where $H \subset G$ is a closed connected subgroup & $\mathcal{L}$ is a multiplicative local system on H.
(new even for U_n).

In particular, any 1-dim rep of $G(\mathbb{F}_q)$ comes from a multiplicative local system on G (new even for algebra groups).

L-packets of irreps of $G(\mathbb{F}_q)$ $\xrightarrow{ⓐ}$ definition in terms of $D_G(G)$ (geometrizing 1)

L-packets of irreps of $G(\mathbb{F}_q)$ $\xrightarrow{ⓑ}$ explicit description/construction in terms of "admissible pairs" for G (geometrizing 2)

POV of $\underline{a}$: L-packets measure failure of $D_G(G)$ to be symmetric, ie nontriviality of square of the braiding on $D_G(G)$

POV of $\underline{b}$: L-packets measure the failure of G to satisfy the main theorem 1'.

Can also relate connectedness of centralizers to the square of the braiding β^2.

This is what allows one to prove Main Theorem 1'.

(Very) Informal Meaning of L-packets

G algebraic group $/\mathbb{F}_q$,

hope for a notion of L-indistinguishability of irreducible representations of $G(\mathbb{F}_q)$ ....

i.e reps that can't be told apart by geometric methods.

L-packets are the resulting equivalence classes.

(for reductive groups $\rightsquigarrow$ Lusztig)

If G is unipotent & connected $/\mathbb{F}_q$

let's say two irreps ρ_1, ρ_2 of $G(\mathbb{F}_q)$ / $\overline{\mathbb{Q}}_\ell$ are L-indistinguishable if

$\forall$ connected closed subgroup $H \subset G$

& any multiplicative local system $\mathcal{L}$ on H

$$[t_{\mathcal{L}} : \rho_1|_{H(\mathbb{F}_q)}] = [t_{\mathcal{L}} : \rho_2|_{H(\mathbb{F}_q)}]$$

Caution: this could be a wrong definition!

Let's go back to geometrizing statement 2:

- remove the words "irreducible representation" from it.

Γ = a finite group

Define $MC(\Gamma)$ = the set of conj. classes of pairs (H, χ) where $H \subset \Gamma$ &

$\chi : H \longrightarrow \mathbb{C}^*$ homomorphism, satisfying

<u>Mackey's criterion</u>:

$$\forall\, \gamma \in \Gamma \setminus H, \quad \chi|_{H \cap H^\gamma} \neq \chi^\gamma|_{H \cap H^\gamma}$$

where $H^\gamma = \gamma^{-1} H \gamma$, $\quad \chi^\gamma(g) = \chi(\gamma g \gamma^{-1})$

If $\mathcal{C} \in MC(\Gamma)$ then $\forall (H, \chi) \in \mathcal{C}$ the induced representation depends only on $\mathcal{C}$ up to isomorphism,

$\rho_{\mathcal{C}} = \mathrm{Ind}_H^\Gamma \chi$, & is irreducible

Thus $MC(\Gamma) \longrightarrow \hat{\Gamma}$.

The fibers of this map are equivalence classes for the relation

$$\mathcal{C}_1 \sim \mathcal{C}_2 \iff \exists\, (H_j, \chi_j) \in \mathcal{C}_j \text{ s.t. } \chi_1|_{H_1 \cap H_2} = \chi_2|_{H_1 \cap H_2}$$

(this follows from Frobenius reciprocity)

Finally, if Γ is nilpotent $\Rightarrow$ $\mathcal{M}(\Gamma) \twoheadrightarrow \hat{\Gamma}$
so we get a nice description of $\hat{\Gamma}$.

Now let's try to generalize this.
Take $k = \bar{k}$ alg. closed, $\ell \neq \operatorname{char} k$,
G connected unipotent group $/k$.

We need to work with G-conjugacy classes of pairs $(H, \mathcal{L})$ where $H \subset G$ is closed & connected, and $\mathcal{L}$ is a multiplicative local system on H.

* Naive analog of Mackey condition:
$$\forall g \in G(k) \setminus H(k), \quad \mathcal{L}|_{(H \cap H^g)^\circ} \not\cong \mathcal{L}^g|_{(H \cap H^g)^\circ}$$

... this is too strong: usually not enough such pairs - e.g. for finite Heisenberg only connected $H \subset G$ are the center, & no local system on the center can satisfy Mackey..

We need to relax this condition
$\rightsquigarrow$ get notion of admissible pair

e.g. G = fake Heisenberg $\Rightarrow$ $\forall$ nontrivial multiplicative local system on $Z(G)$ gives an admissible pair $(Z(G), \mathcal{L})$

$\neq$ [Note: for fake Heisenberg can have multiplicative local systems that restrict nontrivially to the commutator subgroups !]

<u>Definition of an admissible pair</u>

3 conditions in the definition :

[It's possible to define the normalizer of $(H, \mathcal{L})$ as a closed subgroup of G,
$G' \subset N_G(H) \subset G$ ... naively only makes sense at the level of k-points]

1. $(G')^\circ/H$ is commutative

2. the homomorphism $(G'^\circ/H) \to (G'^\circ/H)^*$ induced by $\mathcal{L}$ is an isogeny

3. $\forall g \in G(k) \setminus G'(k)$ we have

$$\mathcal{L}\big|_{(H \cap H^g)^\circ} \not\cong \mathcal{L}^g\big|_{(H \cap H^g)^\circ}$$

About 2: U = any connected commutative unipotent group $/k$, its Serre dual U^* (of the same kind) has $U^*(k) \simeq$ isom classes $= \mathrm{Ext}^1(U, \mathbb{Q}_p/\mathbb{Z}_p)$ of multiplicative local systems on U

Can define dual for any

conn. unipotent group $\to$ possibly disconnected unipotent commutative group

The homomorphism mentioned in 2 is a generalization of the following:

Γ finite, $H \subset \Gamma$ & $\chi : H \to \mathbb{C}^*$ character

Γ' = normalizer of (H, χ) in Γ

χ induces a bimultiplicative map

$$\Gamma'/H \times Z(\Gamma'/H) \longrightarrow \mathbb{C}^*$$
$$(g, \gamma) \longmapsto \chi(g\gamma g^{-1}\gamma^{-1})$$

(... if Γ'/H commutative we get a bilinear form on Γ'/H)

This induces $\Gamma'/H \to (Z(\Gamma'/H))^*$ Pontryagin dual

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Go back to finite fields:

G connected unipotent group $/\mathbb{F}_q$

We say that two pairs $(H_1, \mathcal{L}_1), (H_2, \mathcal{L}_2)$ defined over $\mathbb{F}_q$ are <u>geometrically conjugate</u> if $\exists\, g \in G(\overline{\mathbb{F}}_q)$ s.t. $H_2 = H_1^g$ & $\mathcal{L}_2 = \mathcal{L}_1^g$.

A pair (H, χ) in $G / \mathbb{F}_q$ is admissible if $(H \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \chi \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q)$ is admissible for $G \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$

__Def__ Let $\mathcal{C}$ be a geometric conjugacy class of admissible pairs for G.
Define $L(\mathcal{C}) = \{\rho \in \widehat{G(\mathbb{F}_q)} : \exists (H, \chi) \in \mathcal{C}$ s.t. $\rho|_{H(\mathbb{F}_q)}$ contains t_χ as a direct summand $\}$

i.e. $[t_\chi : \rho|_{H(\mathbb{F}_q)}] > 0$

$\sim$ nonempty! contains __induced__ rep

__Main Theorem 2__

a) If $\mathcal{C}_1, \mathcal{C}_2$ are two geometric conjugacy classes of admissible pairs for G then either $L(\mathcal{C}_1) = L(\mathcal{C}_2)$ or $L(\mathcal{C}_1) \cap L(\mathcal{C}_2) = \emptyset$

b) Every irrep appears in one of these subsets

c) The subsets $L(\mathcal{C})$ are precisely the equiv classes for the correct def. of L-indistinguishability — to appear next time!