CENTRE DE RECHERCHES MATHÉMATIQUES

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Orbit method: well work with unipotent groups

One possible goal of the theory of character sheaves: consider an algebraic group G_0 over $\mathbb{F}_q$, & try to produce a collection $CS(G)$ of irreducible perverse sheaves on $G = G_0 \underset{\mathbb{F}_q}{\otimes} \overline{\mathbb{F}_q}$ which enjoys the following property:

let $Fr : G \longrightarrow G$ denote the Frobenius endomorphism (from absolute Frobenius on G_0)

& let $CS(G)^{Fr} = \{ M \in CS(G) \mid Fr^* M \simeq M \}$

For every $M \in CS(G)^{Fr}$, choose $\psi_M : Fr^* M \xrightarrow{\sim} M$ (any two proportional)

$\Rightarrow$ function $t_M : G_0(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}_\ell}$

by
$$t_M(g) = \sum_{i \in \mathbb{Z}} (-1)^i \, tr(\psi_M, H^i(M_g))$$
$[g \in G_0(\mathbb{F}_q) = G(\mathbb{F}_q)^{Fr}.]$

Property: the irreducible characters of $G_0(\mathbb{F}_q)$ can be "reconstructed" from the t_M's.

Lusztig: solve this problem for reductive groups.

Today: solve for unipotent groups (w/ some restrictions).

Reminder about the orbit method

Γ = finite group, $Fun(\Gamma)$ = functions $\Gamma \longrightarrow \overline{\mathbb{Q}_\ell}$ is an algebra wrt convolution

class functions $Fun(\Gamma)^{\Gamma}$ form subalgebra $= Z(Fun(\Gamma))$

Key point: irreducible characters of Γ are (up to scaling) the same as the minimal idempotents in $\mathrm{Fun}(\Gamma)^\Gamma$.
If we had a Fourier transform making convolution to multiplication easy to read off idempotents.

Orbit method works when $\exists c \in \mathbb{N}$ s.t. $\Gamma \in \mathrm{Nilp}_c$

$= \{$ (possibly infinite) abstract groups N s.t. 1. nilpotence class of N is $\leq c$
2. the map $N \to N$ $x \mapsto x^k$ is bijective for $1 \leq k \leq c$ $\}$

M. Lazard defined isomorphisms of categories

$$\mathrm{Nilp}_c \underset{\mathrm{Exp}}{\overset{\mathrm{Log}}{\rightleftarrows}} \mathrm{nilp}_c = \{ \text{Lie algebras } \mathfrak{n} \text{ over } \mathbb{Z}[\tfrac{1}{c!}] \text{ with nilpotence class } \leq c \}$$

$\mathfrak{n} \in \mathrm{nilp}_c$: $\mathrm{Exp}(\mathfrak{n})$ is $\mathfrak{n}$ as set with multiplication given by Baker–Campbell–Hausdorff
$CH(x,y) = \log(\exp(x)\exp(y))$

Now let $\Gamma \in \mathrm{Nilp}_c$ be finite, $\mathfrak{g} = \mathrm{Log}(\Gamma)$.
& $\exp : \mathfrak{g} \to \Gamma$ the identity map on underlying sets.
Have action of Γ on $\mathfrak{g}$ & on $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{g}, \bar{\mathbb{Q}}_\ell^\times)$

Key result: $\exp^* : (\mathrm{Fun}\,\Gamma)^\Gamma \longrightarrow (\mathrm{Fun}\,\mathfrak{g})^\Gamma$
is an isomorphism of algebras (RHS: convolution for addition).

Easy: Fourier transform $\mathcal{F} : \mathrm{Fun}(\mathfrak{g}) \to \mathrm{Fun}(\mathfrak{g}^*)$

$$\mathcal{F}(f)(\lambda) = \sum_{x \in \mathfrak{g}} \lambda(x) f(x)$$ is a Γ-equivariant

isomorphism of algebras $(\mathrm{Fun}\,\mathfrak{g}, *) \longrightarrow (\mathrm{Fun}\,\mathfrak{g}^*, \cdot)$

$$\Rightarrow ((\mathrm{Fun}\,\Gamma)^{\Gamma}, *) \simeq (\mathrm{Fun}\,\mathfrak{g}^*, \cdot)$$

So irreducible characters of Γ $\longleftrightarrow$ Γ-orbits on $\mathfrak{g}^*$,
ie Kirillov character formula: irred characters are Fourier transforms of coadjoint orbits.

<u>Geometric setting</u>

Γ = finite nilpotent group $\longleftrightarrow$ G = unipotent group over $k = \bar{k}$, char $k = p \neq \ell$

$\Gamma \in \mathrm{Nil}_{p_c}$ for some $c \in \mathbb{N}$ $\longleftrightarrow$ Nilpotence class of G is $< p$

$\mathrm{Fun}(\Gamma)$ $\longleftrightarrow$ $\mathcal{D}(G) = D^b_c(G, \overline{\mathbb{Q}}_\ell)$ constructible derived category

$\mathrm{Fun}(\Gamma)^{\Gamma}$ $\longleftrightarrow$ $\mathcal{D}_G(G)$ = G-equivariant objects in $\mathcal{D}(G)$

convolution $*$ $\longleftrightarrow$ convolution: let $\mu, p_1, p_2 : G \times G \longrightarrow G$
$M * N := \mu_!(p_1^* M \otimes p_2^* N)$

$\mathfrak{g} = \mathrm{Log}\,\Gamma$ $\longleftrightarrow$ $\mathfrak{g} = \mathrm{Log}\,G$ is the Lie ring scheme representing the functor
$k\text{-schemes} \longrightarrow \mathrm{nilp}_{p-1}$, $S \longmapsto \mathrm{Log}(G(S))$
(won't in general come from a Lie algebra!)

Remark: even if $G^p = 1$, $\mathfrak{g}$ may NOT come from a Lie algebra over k (as a set functor it is the same as G)
For example if $p \geq 3$ $\exists$ <u>noncommutative</u> 2-dimensional G's so can't come from a Lie algebra.

Pontrjagin dual $\longleftrightarrow$ Serre dual:

Given a commutative unipotent group $\mathfrak{g}$ over k, connected, $\exists$ another connected comm. unipotent $\mathfrak{g}'$ and a central extension

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathcal{E} \to \mathfrak{g}' \times \mathfrak{g} \to 0$$

s.t.

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathcal{E}^{perf} \to \mathfrak{g}'^{perf} \times \mathfrak{g}^{perf} \to 0$$

is a universal central extension of $\mathfrak{g}^{perf}$ by $\mathbb{Q}_p/\mathbb{Z}_p$.

Fix an isomorphism $\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} \mu_{p^\infty} \subset \overline{\mathbb{Q}}_\ell^\times$ & let $\mathcal{L}$ denote the local system on $\mathfrak{g}' \times \mathfrak{g}$ induced from $\mathcal{E}$ via this.

Fourier-transform $\longleftrightarrow$ Fourier-Deligne transform

$$\mathcal{F} : \mathcal{D}(\mathfrak{g}) \longrightarrow \mathcal{D}(\mathfrak{g}')$$

$$\mathcal{F}(M) = pr'_!\left((pr^* M) \otimes \mathcal{L}\right)[d]$$

$d = \dim \mathfrak{g}$. takes $\star$ to $\otimes$.

Key properties! i) $\mathcal{F}$ preserves perverse sheaves

ii) $\mathcal{F}(M \star N) = \mathcal{F}(M) \otimes \mathcal{F}(N))[d]$

If $\mathfrak{g} = \mathrm{Log}\, G \Rightarrow$

$$\mathcal{F} : \mathcal{D}_G(\mathfrak{g}) \xrightarrow{\sim} \mathcal{D}_G(\mathfrak{g}')$$

Remark: all of this makes sense for $k = \overline{\mathbb{F}}_q$ & in this case $t_\mathcal{L} : \mathfrak{g}'(\mathbb{F}_q) \times \mathfrak{g}(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$

is a perfect pairing.

Analogs of minimal idempotents $\longleftrightarrow$ char functions of coadjoint orbits: should be $i_{\Omega !} \overline{\mathbb{Q}}_\ell$ constant sheaf on orbit $\Omega \subset \mathfrak{g}'$

(coadjoint orbits are closed for unipotent groups)

Convenient to remember $\iota_\ell : \overline{\mathbb{Q}}_\ell \longleftarrow \overline{\mathbb{Q}}_\ell$ from adjunction

Def An object e in a monoidal category $\mathcal{M}$ is an idempotent if $\exists\ \mathbb{1} \longrightarrow e$ which becomes an isomorphism after tensoring by e on either side.

Naive guess: if $k = \overline{\mathbb{F}}_q$ & $G = G_0 \otimes_{\mathbb{F}_q} k$ the character sheaves on G should be minimal idempotents in $\mathcal{D}_G(G)$.

... not good enough in general, because if $\Omega \subset \mathfrak{g}'$ is defined over $\mathbb{F}_q$ then $\Omega(\mathbb{F}_q)$ may not be a single $G(\mathbb{F}_q)$-orbit.

Theorem Assume that the G-stabilizers of points in $\mathfrak{g}'$ are connected. For every minimal idempotent $e \in \mathcal{D}_G(G)$ there exists $0 \le n_e \le \dim G$ s.t. $e[-n_e]$ is perverse, & if $Fr^*(e) \cong e$ there is a canonical choice of $\psi_e : Fr^*(e) \to e$ s.t. $t_e\, q^{\frac{d+n_e}{2}}$ is an irreducible character of $G_0(\mathbb{F}_q)$ & every irred character is obtained this way.

General setting (nilpotence class $< p$): replace minimal idempotents:
$e \in \mathcal{D}_G(G)$ minimal idempotent

$$\{ M \in \mathcal{D}_G(G) \mid e * M \cong M \}$$

$\cup$

$\mathcal{M}_e^{perv}$ = full subcat. of perverse sheaves

Def: a character sheaf for e is an indecomposable object of $\mathcal{M}_e^{perv}$.

Theorem: this construction achieves desired goals.
& $\mathcal{M}_e^{perv}$ is semisimple. Closedness of orbits $\longleftrightarrow$ idempotency.