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Langlands Seminar
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Digression [Beilinson: superscheme $(X, \mathcal{O}^{ev} \oplus \mathcal{O}^{odd})$ gives scheme in two different ways: subscheme $(X, \mathcal{O}^{\cdot}/\langle \mathcal{O}^{odd} \rangle)$ & quotient $(X, \mathcal{O}^{ev})$.]

Derived algebraic geometry: $(X, \mathcal{O}_X^{\cdot})$ scheme with sheaf of dg commutative algebra
Locally $X$ is $\simeq$ Spec $H_{\cdot}(\mathcal{O}_X^{\cdot})$ topologically

Stable homotopy category looks like dg algebra after tensoring with $\mathbb{Q}$, but very different before tensoring.

Example: $X$ scheme $K_{\cdot}^{alg}(\mathcal{O}_X)$ sheaf of algebraic $K$-groups $= \pi_* BGL^+ \mathcal{O}_X$ $K_0(R)$ should pass to homotopy groups only at last stage, work with space (or object of stable homotopy category) itself.
$K$-theory: algebra (or space) $\longmapsto$ spectrum, can be easier to describe than its homotopy groups!

<u>Spectra</u> $\left(\begin{array}{c}\text{Pointed spaces} \\ \text{..eg CW complexes}\end{array}\right) \longrightarrow$ Spectra $\overset{?}{\sim}$ abelian group objects approx in category of spaces

$\pi_n(X) = [S^n, X]$

Ring spectra $\sim$ Ring objects

Abelian group objects? e.g. infinite loop space: sequence of spaces $X_n$ + isomorphism $X_n \xrightarrow{\sim} \Omega X_{n+1}$
Eilenberg-MacLane spaces $H(A,n)$, $\pi_i H(A,n) = \begin{cases} A & n \\ 0 & \text{otherwise} \end{cases}$

$\pi_{i+1} X \simeq \pi_i \Omega X$
$\quad \| \qquad\qquad \|$
$\mathrm{Map}(S^{i+1}, X) \quad \mathrm{Map}(S^i, \Omega X)$

FACT The abelian group objects in the homotopy category are products of Eilenberg-MacLane spaces!

Spectra $\longleftrightarrow$ $\mathbb{S}$-modules: modules over sphere spectrum — analog of abelian groups = $\mathbb{Z}$-modules in usual world.

Products! On the other hand a commutative top group is homotopy equivalent to a product of E-McL spaces.

— shouldn't think of locally compact spaces, but compactly generated ones — eg realizations of simplicial sets.

Simplicial abelian groups $\xleftrightarrow{\text{Dold-Puppe}}$ Chain complexes (non pos graded)

Each such complex is (noncanonically) isomorphic to $\oplus$ of its cohomologies ... ie E-McL spaces...
~~upto~~ ... since all $\mathrm{Ext}^i$ $(i \geq 2)$ vanish!
... not true more generally

Spectra form symmetric monoidal category, under smash...
What do we mean by a commutative ring spectrum?
Naively: rings in this symmetric monoidal category
Bad idea, even in world of complexes!
want dga up to quisomorphism, rather than commutative ring in homotopy category of complexes.

Symmetric group action:
$E\Sigma_n \times_{\Sigma_n} R \wedge \dots \wedge R \longrightarrow R$ commutativity data.

Two notions: 1. commutative dga
2. associative dga which are homotopy commutative!
e.g. $ab \neq ba$ homotopic, + higher compatibilities $\Rightarrow$ $E_\infty$ algebra
— same after $\otimes \mathbb{Q}$, different in $\neq 0$ characteristic.
Here addition is commutative on the nose.
but could replace this also by something more homotopic $\Rightarrow$ commutative ring spectra
ie spectrum $\longleftrightarrow$ weakened version of ~~comm dga~~ $E_\infty$ gps
commutative group.
$\Rightarrow$ $E_\infty$ ring spectra

Remark $Q(X) := \lim_{n\to\infty} \Omega^n S^n X \sim \bigcup_{n\geq 0} E\Sigma_n \times_{\Sigma_n} X^n / \sim$

$\otimes \mathbb{Q}$ : $SP^\infty(X) = \bigvee X^n/\Sigma_n/\sim$ (basepoints)

Since $E\Sigma_n/\Sigma_n$ purely torsion, vanishes $\otimes \mathbb{Q}$.

But homotopy theoretically very different.

$\pi_* Q(X) = \pi_*^S(X)$ ... primitives in homology

tensor algebra on $H_*(X.\mathbb{Q})$ $= H_*(SP^\infty X, \mathbb{Q})$

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Enriched scheme (Lurie, Toen-Vezzosi) $\sim$

$X$ space, in Zariski topology have sheaf $\tilde{\mathcal{O}}_X$ of $E_\infty$ ring spectra (comm S-algebra)

Brave new = functor from old to new.

$X$ locally homeo to Spec $\pi_0 \tilde{\mathcal{O}}_X$.

$\mathbb{S}$ = Brave new $(\mathbb{Z})$. Good old world $\subset$ brave new world

$1 \in \{S^0 \to H\mathbb{Z}\} = \pi_0(\mathbb{Z}) = \mathbb{Z}$
$= \tilde{H}^0(S^0, \mathbb{Z})$

$\Rightarrow$ map $S \to \mathbb{Z}$ of ~~ring~~ $E_\infty$ ring spectra.

Brave new-module over $\mathbb{Z}$ is a complex of abelian groups

$H : D(\mathbb{Z}\text{-mod}) \to (S^0\text{-Mod})$

Spec $\mathbb{Z} \xrightarrow{i}$ Spec "S" enriched scheme

modules over Spec $S$ is just a spectrum.

Spec $\mathbb{Z} \to$ Spec $S$ far from flat — so can make any scheme into an enriched scheme via Eilenberg-MacLane, but not in flat way...

Quillen: $\pi_* MU \underset{\text{canonical}}{\cong}$ Lazard's ring classifying 1-dim formal group laws

$\pi_* MU$ represents functor of formal group laws / $A$
$MU_* MU = \pi_*(MU \wedge MU)$ represents ~~func~~ groupoid of pairs of formal groups + isom between them.

$S^0 \to MU \rightrightarrows MU \wedge MU \rightrightarrows MU \wedge MU \wedge MU \cdots$
Adams resolution of $S^0$:
totalize get something homotopic to $S^0$, filter get AMSS
Take Spec: FG Laws $\leftleftarrows$ isomorphisms between formal groups $\leftleftarrows$ ⋯
Simplicial realization of stack of formal groups.

AMSS : $E_2 = \mathrm{Ext}^*_{fgps/G}(\mathcal{O}_L, \mathcal{O}_L) \Rightarrow G_r[S^0, S^0]$.

So want to calculate $H^*_G(L, \mathcal{O}_L)$
$L =$ ~~space~~ of FGLs
$G =$ automorphisms of formal groups. need cohomology in structure sheaf, which doesn't satisfy excision ~~both~~ no unlike cohomology with constant coefficients.

$MU_* MU = \pi_* MU \otimes S_*$
$S_* =$ Landweber - Novikov algebra
$\mathrm{Spec}\ S_* = \mathrm{Diff}(\hat{A}_0)$

$S_*$ is not $\pi_*$ of a natural spectrum.
Groupoid action doesn't split as a product in category of spectra --- so not a brave new group quotient.

Spaces, spectra are triangulated categories with $\otimes$
$\Rightarrow$ consider functors to Vect

eg space $X \mapsto H^*(X, \mathbb{F}_p)$. Monoidal functor
$X \wedge Y \mapsto H^*(X, \mathbb{F}_p) \otimes H^*(Y, \mathbb{F}_p)$

What are automorphisms (multiplicative) of this functor $H$?
(ie commute with Künneth)

Vary ring of coefficients $\Rightarrow$ functor $\mathrm{Aut}(H)$ ($\phi$ —) $\in$ Groups
- representable by algebraic group (eg Steenrod algebra)

Adams spectral sequences: $H^*(\mathrm{Aut}(F), \mathrm{Ext}()) \Rightarrow [X, Y]$

~~So~~ — these functors in topology are not faithful
but get spectral sequence from Exts over these automorphisms
to homotopy groups.