J. Morava II - Enriched schemes, spectral sequences of Adams/Leray/Descent type, etc... 10/21/04

1. Concrete statements on homotopy groups

Bott: $\pi_* BU = \begin{cases} \mathbb{Z} & * = \text{even} \\ 0 & * = \text{odd} \end{cases}$

$\Leftrightarrow$ $\pi_* U = \begin{cases} \mathbb{Z} & * = \text{odd} \\ 0 & * = \text{even} \end{cases}$ $U = \lim_{n\to\infty} U(n)$

$\mathbb{Z}[\frac{1}{2}] \otimes \pi_* SO = \mathbb{Z}[\frac{1}{2}]\ * \in 4\mathbb{Z}-1$
$= 0$ else

(ignore 2-torsion!) "super"world

"first picoseconds after big bang all force were united"

$\Leftrightarrow$ first few primes all entangled together, things get better for higher primes

Whitehead: X space "free" ∞ loop space on X is $\Omega^n S^n X = QX$

$\pi_* QX = \lim_{n\to\infty} \pi_{i+n}(S^n \wedge X) = \pi_*^S(X)$ --> this is a cohomology theory

$\shortparallel$

infinite symmetric product on X, rationally

$\exists$ maps $SO(n) \to \Omega^n S^n \Rightarrow J: \pi_* SO \to \pi^S(pt)$

$J: \pi_* SO \longrightarrow \pi_*^S(pt)$ J homomorphism

$\shortparallel$

$\mathbb{Z}$ $* = 4k-1$ image in dim $4k-1$ is cyclic, ~~of order~~ generated by $b_{2k}/2k \in \mathbb{Q}/\mathbb{Z}$

$b_{2k} \leftrightarrow \zeta(1-2k)$ Bernoulli number.

$S^0 \to E$ unital ring spectrum $\Rightarrow$ Hurewicz map
$\pi_*^S \longrightarrow E_*$

e.g. $E = K_*^{alg}(\mathbb{Z})$: get composition
$$\pi_* SO \longrightarrow \pi_*^S(pt) \xrightarrow[\text{Hurewicz}]{} K_*(\mathbb{Z})$$

composition more or less detects $\zeta(1-2k)$,
[not $K_{4k-1}(\mathbb{Z})$, related to $\zeta(2k)$]

$$\mathrm{im}\, J \hookrightarrow K_*(\mathbb{Z}) \text{ in these degrees}$$
$$\downarrow$$
$$K_*(\mathbb{F}_\ell) \quad \text{computed by Quillen} \quad (\ell, p) = 1$$

[Note: Spectra are additive category, Homs $\in Ab$,
Spectra$_{(p)}$ has Hom in $\mathbb{Z}_{(p)}$-mod]

Work p-locally
Classical fact write $k = 2p^{\nu}(p-1)\cdot$ ___
Then $\zeta(1-2k) = p^{-(\nu+1)} \in \mathbb{Q}/\mathbb{Z}^{(p)}$ (von Staudt - Clausen thm)
So p-component of Bernoulli numbers have logic,
can understand p-part of J-homomorphism

2. Adams S Seq, in particular for K-theory

(Spectra)$_{(p)}$ is a triangulated symmetric monoidal category
with duals for finite objects.

$$H_*(-, \mathbb{F}_p) : (\text{Spectra})_{(p)} \longrightarrow \mathbb{F}_p\text{-Vect} \quad \left(\begin{array}{c}\text{really} \\ \underline{\text{super}} \\ \text{graded}\end{array}\right)$$

1. $\exists$ Künneth theorem: $H_*(X \wedge Y) = H_*(X) \otimes H_*(Y)$
$\Rightarrow$ monoidal functor

... homology of a spectrum: $\lim_{n\to\infty} H_{i+n}(X_n) = H_*(X)$

= primitives in $H^*(\Omega^\infty X) \otimes \mathbb{Q}$

2. Not faithful, But does reflect isomorphisms!
... conservative: If $H_*(f)$ is iso, then
f is a homotopy equivalence.

Consider multiplicative automorphisms of such a functor
$\alpha : H_*(-) \circlearrowleft$ preserving Künneth isomorphism

$\Rightarrow$ $\mathrm{Aut}(H\mathbb{F}_p)$ is a pro-algebraic group,
representing a Hopf algebra, dual to classical
Steenrod algebra of reduced powers
(well also have super part!
$A_* =$ poly $\otimes$ exterior, we're discussing polynomial..
... have super part)

So can regard $H\mathbb{F}_p : (\mathrm{Spectra})_{(p)} \longrightarrow$ Reps of this proalg. group

((Classically $[H\mathbb{F}_p, H\mathbb{F}_p]_* = \mathrm{End}(H\mathbb{F}_p)_* = A^*$)

Adams SSeq:

$$E_2 = \mathrm{Ext}_{\mathrm{Aut}\,H\text{-Reps}}(H_*(X), H_*(Y)) \Rightarrow [X, Y]_{(p)}$$

convergence comes from conservativity of $H\mathbb{F}_p$.

To get SSeq: build resolution of S^0 by $\underline{H\mathbb{F}_p}$-free objects

You can try this for any cohomology theory!

Ex. $K_{\mathbb{C}}$ = complex K-theory
work p-completely $\Rightarrow$ $K_p^\wedge$ p-adically completed

$K^*(X)_p^\wedge = \varprojlim K^*(X_i) \otimes \hat{\mathbb{Z}}_p$ limit over finite subcomplexes, $\leftarrow$ compact $\mathbb{Z}_p$-modules

Have Künneth theorem
$\mathrm{Aut}(\hat{K}_p) \Rightarrow$ Adams operations $\psi^k : K \circlearrowleft$ multiplicative

... eg G finite $\mathrm{Rep}_{\mathbb{C}}(G) \leftrightarrow \chi : \hat{G} \to \mathbb{C}$
$\psi^k \chi(\hat{g}) = \chi(\hat{g}^k)$

natural operations, k-th power
$\Rightarrow \mathrm{End}(\mathrm{Rep}_{\mathbb{C}}(-)) = \mathbb{Z}^\times$ as a monoid
(for $(-)$ finite groups.

Lemma (Atiyah?) $\psi^{k+\ell}(x) - \psi^k(x)$ highly divisible by p if ℓ is $\Rightarrow$
p-adically continuous

$\Rightarrow \psi^\alpha$ defined in $K_p^\wedge$ for $\forall \alpha \in \mathbb{Z}_p$.

- Natural multiplicative transformations of $K_p^\wedge = \hat{\mathbb{Z}}_p$

$\mathrm{Aut}\ K_p^\wedge =\cdot \hat{\mathbb{Z}}_p^\times \longleftrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)^{ab}$, Quillen, Sullivan exploit this

e.g. $K(S^{2n}) = \mathbb{Z} \cdot b^n$
$\psi^k(b) = kb$, $\psi^k(b^n) = k^n b^n$, $\psi^\alpha(b^n) = \alpha^n b^n$?

$K_p^\wedge(S^{2n}) = \hat{\mathbb{Z}}_p b^n$ $\alpha \in \hat{\mathbb{Z}}_p^\times$ acts by $b^n \mapsto \alpha^n b^n$

... "Bott motive", analog of Tate motive $\mathbb{Z}(n)$

Unstable operation $\psi^P(b) = p \cdot b$, not invertible $\notin \mathbb{Z}_p^*$
Can't be made to commute strictly w/ suspension.

$\leftrightarrow$ related to isogeny $p: \mathbb{G}_m \to \mathbb{G}_m$.
isogenies give unstable operations.

$\exists$ Adams SSeq for $K_p^\wedge$:

$\mathrm{Ext}^k_{\hat{\mathbb{Z}}_p^*\text{-mod}}(K(S^0)_p^\wedge, K(S^{2n}_p)^\wedge)$, doesn't converge to $\pi_{2n-k}(S^0)_p^\wedge$

$\mathrm{Ext}^k = 0$ if $k > 1$ $\Rightarrow$ only two lines!

$k=1$: $H^1(\hat{\mathbb{Z}}_p^*, \underbrace{\hat{\mathbb{Z}}_p(n)}_{= M})$ $\alpha \in \hat{\mathbb{Z}}_p^*$ acts as mult by α^n.

$\hat{\mathbb{Z}}_p^*$ topologically cyclic: generated by g,
$g \equiv 1 \bmod p$, $\not\equiv 1 \bmod p^2$

$$H^1(\hat{\mathbb{Z}}_p^*, M) = M/(g-1)M \qquad M = \hat{\mathbb{Z}}_p(n)$$
$$= \mathbb{Z}/p^{\nu+1}\mathbb{Z} \quad \text{when } n = 2p^\nu(p-1)\cdot \_$$

SSeq converges to Im J.

Most geometric constructions involving stable homotopy theory are built out of Im J.

On the other hand Milnor-Kervaire group Θ_* of smoothings of the sphere are

$$1 \to \text{Bernoulli part} \to \Theta_* \to \mathrm{Coker}\, J \to 0$$

Bernoulli part $=$ Geometrically ~~[illegible]~~ constructible spheres e.g. Brieskorn (parallelizable) ... ~~Ber~~ Bernoulli numerators

3. $S^0 \to \mathbb{Z}$

E cohomology theory has a kernel : $\{X \mid E_*(X)=0\}$
($X \in$ finite spectra)

$\Rightarrow$ can localize : Bousfield localization ... ~~natural~~
~~hence~~ $E \mapsto$ subcategory of E-local spectra,
Localization is best approximation by E-local spectra, $L_E X$

Naive Hope : an Adams Spectral Seq for E-spectra will converge to $[L_E X, L_E Y]$.

Moduli stack of formal groups Lazard/Diff :
want map Lazard/Diff $\to$ "Spec S^0"

Want E-local spectra to live over affine patch in Spec S^0
e.g. A commutative ring $\Rightarrow$ localizing subcategories of $D(A\text{-mod})$ (Δ subcats closed under direct summands Δ ∞ direct sums)
give subvarieties
recover spectrum from thick subcategories of $D(A\text{-mod})$

... want to make "Spectrum of category of spectra", using thick subcategories of category of spectra

Want Spectra ~ D(scheme) : both $\otimes \Delta$ categories ...
with infinite direct sums, compactly generated
-- spectra in fact generated by unit elements!
$\Rightarrow$ compact objects have duals.

Closed subset $\Rightarrow$ full subcategory with cohomologies concentrated there, "an ideal" in $D(X)$: stable under $\otimes$ by any object
In topology: any Δ subcategory stable under $\oplus$ is $\otimes$ stable, since generated by 1 Δ anything stable by $1 \otimes$ _

Adjoint functor to quotient by such localizing subcategory gives subcategory of (E)-local objects.

{ (Localizing subcategories) $\cap$ (finite objects) }

$\Updownarrow$

Hopkins-Devinatz-Smith

{ isom classes of 1-dim formal group laws }

Closed substacks of 1-dim formal groups $\longleftrightarrow$ compactly generated Δ subcategories closed under $\otimes$ of spectra (p locally).

$\longleftrightarrow \{p^n\}_{1 \le n \le \infty}$ & 0

Theorem (Hopkins Smith) X finite connected spectrum.
$End_* X$ = graded ring of selfmaps $[S^* X, X]$ (suspension) has a nontrivial center of Krull dim 1, either $\mathbb{Z}_{(p)}$ or $\mathbb{F}_p[w]$, $|w| = ?(p^n - 1)$.

$S^{(p^n-1)} X \xrightarrow{w} X$ central, take cofiber
$\to$ cofiber of w $\ni$ self-maps of higher order.

- get higher & higher pieces
$\Rightarrow$ whole category of spectra has ∞ Krull dimension:

generic point

$\cdot 2^1 \to 2^2 \to 2^3 \to \cdots \to 2^\infty$

3

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$7^1 \to 7^2 \to 7^3 \cdots \to 7^\infty$

K-theory

Spec $\mathbb{Z}$ closed point

$\updownarrow$

ordinary cohomology

$\updownarrow$

$\mathbb{C}_\infty$ infinite height

K-theory, H $\in$ Lazard / Diff $\longrightarrow$ Spec S^0

... fiber functors for spectra

H at bottom of hierarchy, sees everything. As go further down see more & more

4. Other examples of enriched schemes

Elliptic cohomology: Stack of elliptic curves $Ell \longrightarrow \mathcal{M}_{fg}$

Digression $\mathcal{M}_{fg}$ stack $\longrightarrow$ "Spec S^0" : Coarse moduli of formal groups

no

$G \circlearrowright X \Rightarrow [X/G]$ as groupoid

$\Rightarrow X/G$ as space

$\cdot \to \cdot \to \cdot \to \cdots \to \cdot$

Spec S^0 ringed space, not quite a scheme ... tree: looks like iterated discrete valuation ring

K open in Spec S^0, take localization $L_K S^0 \Rightarrow$ E_∞ ring spectrum, get good sheaf, global sections is naive Spec S^0

$\exists$ cohomology theory taking values in category of sheaves / Ell.

$\exists$ line bundle over moduli Ell, $\bigoplus \Gamma \mathcal{L}^{\otimes k}$ algebra of topological fm

$\mathcal{L} = \mathcal{E}(S^2)$ $\quad \mathcal{L}^{\otimes k} = \mathcal{E}(S^{2k})$

$\Gamma \mathcal{L}^{\otimes k} = \oplus \Gamma \mathcal{E}(S^{2k}) = Ell_*(pt)$

I - saved $\exists R^* \Gamma$

$\pi_* S^0 \longrightarrow \pi_* Ell$ Hurewicz

$\pi_*^s \longrightarrow tmf$ is close to injective up to dim 255 (at 2, 3).

<u>Other examples</u>

Possible analog involving K3 surfaces:

Artin-Mazur: formal Brauer group ~ $H^2(X, \hat{\mathbb{G}}_m)$

1-dim formal group for K3s.

-- analog of $\widehat{Pic}\, E$ for E elliptic curve

... height is between 1 & 10 or ∞.

$\exists$ sheaf of cohomology theories over stacks of K3s?

For at least some moduli varieties, stratification by height of Brauer group is given by regular sequence $\Rightarrow$ construct good cohomology theory

Drinfeld Moduli of abelian varieties + endos + polarizations are Shimura varieties, which we understand.

Moduli of K3s is an exotic example of a Shimura variety ~~with no direct moduli defn.~~

- have Shimura varieties without a natural moduli construction, can show have models over small fields, no direct natural approach.

K3s example of Shimura variety with unusual definition

Deligne proof of Weil for K3s: use fact that moduli of K3s is a Shimura variety, sits inside moduli of abelian varieties in strange way, huge dimension A_g, ~~[illegible]~~ specified by algebraically unknown conditions (not PEL type)

— require certain classes to be Hodge:

Deligne – prove Weil for without knowing equations.

K3 $X \longrightarrow H^2(X, \mathbb{Z})$ lattice of dim 21 with orthogonal form. $\rightarrow$ point of Shimura variety for $SO(2,19)$ Hodge moduli space.

sits inside symplectic Shimura variety using Spinor rep, of dim $\sim 2^{20/2}$

This is a transcendental construction, only a posteriori show it is in fact algebraic.

Algebraic POV: piece of formal group for this abelian variety constructible algebraically

? $H^2(X \quad)$ sits inside H^2 of the abelian variety, sits in endos of Clifford module ?