J. Morava – Interview with Beilinson–Drinfeld 2/29/00

Complex bordism is graded, Landweber–Novikov acts (diff of formal line)

splitting grading: incorporates G_m providing grading

even-odd always preserved, anticommutative. Supersymmetries?

don't seem to act. Forget grading full diffeos act..

need more care for infinite complexes.

Finite complex ⇝ O-mod on stack of formal groups

so get module over any ring R from formal gp over R

– kind of cohomology theory for any formal group...

Usual convention: $K(n)^*(-, \mathbb{F}_p)$ cohomology theory, $\mathbb{F}_p$-mod

$K(n)^*(\mathbb{C}P_\infty, \mathbb{F}_q)$: $[p](\zeta) = \zeta^q$, $q = p^n$

height n. No good coord-free construction.

Ref: U. Würgler, Conf. in Poznan, LNM, ~1994?

Question about multiplicative structure --- trouble at $p=2$

$n=1$ $K(1)^* \sim$ essentially K-theory with $\mathbb{F}_p$ coeffs —

NO commutative multiplication at prime 2

— Commutator $[x,y] \sim \gamma\beta(xy)$ Bockstein.

Same for complex cobordism – have noncommutative multiplication.

$MU(\text{space})$ is module over $\mathcal{O}$[mod. of formal grps]

no ring structure etc...?

Product in MU : $MU^*(X)$ an algebra : $V \times_X W \longrightarrow X$

$V \xrightarrow{f} X$ $W \xrightarrow{g} X$ maps ⇒ take

$$\begin{array}{ccc} V\times_X W & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ V\times W & \longrightarrow & X\times X \end{array}$$

Other picture $[X, MU] = \lim_{k\to\infty} [S^{n+k} \wedge X, MU(k)]$

stabilized picture

$$\begin{array}{ccc} BU(k) \times BU(l) & \longrightarrow & BU(k+l) \\ \nearrow & & \nearrow \\ \xi_k \times \xi_l & \longrightarrow & \xi_{k+l} \end{array}$$ Universal bundle

On Thom complexes $BU(k)^{\xi_k} \wedge BU(l)^{\xi_l} \longrightarrow BU(k+l)^{\xi_{k+l}}$

$MU(k) \wedge MU(l) \longrightarrow MU(k+l)$

ring spectrum — get product on $[X, MU]$

(everything on pointed spaces..

Suspension = join with circle, join & suspension don't commute

Künneth: bordism of join more or less tensor product,

up to higher terms

Specialize: naively hope to get commutative ones

Want $H^*(pt)$ comm ring for any formal group, $H^*(-)$ modules over this ring. — too naive since rings aren't commutative... what happens to tensor product as.:
ring of constants always commutative ...
$K(1)(\mathbb{R}P^2, \mathbb{F}_2)$ source of all difficulty (K-theory)

13 (graded) commutative up to two torsion
relates in alg K-theory to $\{-1,-1\} \neq 0$ ...
x^2 need not be zero for x odd : is it almost supercommutative?
$H^*(\mathbb{R}P^n, \mathbb{F}_2)$ is commutative not exterior - ?

- problem is more severe than this ...
$xy - yx = \beta(\text{something})$, $\beta^2 = 0$ $\Rightarrow$
$xy + yx = \beta(xy)$ makes sense.

Q is interaction of involution on $MU(X \times X)$ & product description $MU(x) \otimes MU(x)$ ...
M. Boardman : Hopf algebra of operations mod 2.

L Lazard ring $V \hookrightarrow \operatorname{Spec} L$ subvariety.
$X \mapsto \partial_X^*$ sheaf $MU^*(X)$ of MU^*-modules

$$\begin{array}{ccc} i^*\partial_X^\circ & \dashrightarrow & \partial_X^\circ \\ \downarrow & & \downarrow \\ V & \xrightarrow{i} & \operatorname{Spec} L \end{array}$$

how to construct such pullbacks? Usual pullback doesn't preserve exactness -- won't be cohomology $\dashrightarrow$ need <u>derived</u> pullback $R i^* \partial_X^\circ$
$\exists$ construction when V is complete intersection or open (loc preserves exactness) -- have spectrum realizing the theory
Strata (orbits of operations) <u>are</u> complete intersections

$[p](T) = pT + \dots + v_1 T^p + \dots$ p-series of formal group
Strata at p defined by $v_1, v_2, \dots, v_{n-1} = 0$.

$\underline{MU} \in (\text{Spectra})$, $\exists$ morphisms $v_1, \dots v_{n-1} : \underline{MU} \circlearrowleft$
(of positive degree)
$\rightarrow$ take cokernel : hard in homotopy theory
(need EKMM etc...)

$$\begin{array}{ccc} \underline{MU} & \xrightarrow{v_1} & \underline{MU} \\ {\scriptstyle v_2}\downarrow & & \downarrow{\scriptstyle v_2} \\ \underline{MU} & \xrightarrow{v_1} & \underline{MU} \end{array}$$

take some kind of cone.

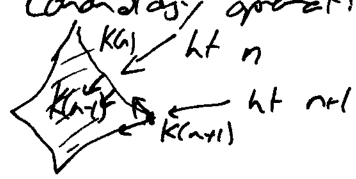
Originally done geometrically, in terms of cobordism with singularities, use Koszul complex of complete singularity....
- but need to choose explicit geometric representatives
see a posteriori indep of choices.
Singularities – corner-type problems.
Same difficulty as with multiplication at $p=2$.
$V \times V$ ↻ flip map, make into mapping torus over S^1
dim V even, torus is odd dim, $MU^{odd}=0$
$\Rightarrow$ torus is the boundary of something
$\underline{MU} \xrightarrow{V} \underline{MU} \longrightarrow$ cofiber

Homotopy of cofiber are $MU^*(pt)/([V])$
$[X, \text{cofiber}]_n = MU/([V])^*(X)$: is this an $MU^*/([V])$-module?
obviously MU^* module, i.e. does mult. by $[V]=0$ "universally"?
Obstruction to this is class of mapping torus...

X, Y Cobordism classes with singularities of type $V_1, \dots V_n$,
$[\partial M = \coprod_k \text{Cone on } V_k \times (-)]$

$Y \times X$ is cobord class with sing of type $\overbrace{V_1, \dots V_n}^{\text{green}}, \overbrace{V_1, \dots V_n}^{\text{blue}}$
But this theory is $MU^*[\underbrace{V_1, \dots V_n}_{\text{green}}] \otimes$ exterior alg $(\underbrace{\beta V_1, \dots \beta V_n}_{\text{blue Bocksteins}})$

$\Rightarrow$ get multiplication, but very tricky!
Cohomology operations essentially same.



$\exists$ universal deformation in normal direction
~ "connection"

at 2 Steenrod algebra: formal distributions on formal additive group.
In general Steenrod coalgebra (dual)
$$A^* = \mathbb{F}_2[\xi_1, \xi_2 \dots] \text{ polynomial}, \quad \Delta \xi_k = \sum \xi_{k-i}^{p^i} \otimes \xi_i$$
$$x \longmapsto \sum \xi_i x^{2^i}$$
Spec A^*
p odd $A^* = P(\xi_1, \dots) \otimes E(\tau_1, \dots)$ poly $\otimes$ exterior
Milnor's Bocksteins
Milnor β's : $Q_k : H^* \longrightarrow H^{*+2p^k-1}$, derivations.

$P(\xi, \ldots) \longleftrightarrow \mathrm{Aut}(\hat{G}_a)$ $E^{\cdot}(N)$
$E(\tau, \ldots) \longleftrightarrow$ exterior algebra on normal sheaf to $\hat{G}_a$
in moduli of formal groups (codimension ∞)
$N = H^2_s(\hat{G}_a, \Pi_p)$ Lubin–Tate deformation group

Bocksteins come as normal directions to each $K(n+1)$
along $K(n)$ ...
$\partial M = \coprod$ Cone V_{α} $\times (Q_0, n)$ Böcksteins
$\sim Q_i^2 = 0$: just piece of singularity of ∂M.

Have coaction $E^*(N) \otimes$ module $\to$ module
∇_{normal} normal connection/derivation, natural under automorphisms
$A_{(2)}$ noncommutative, extension of exterior by something with polynomial

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$MU(-)$ triangulated functor from stable homotopy category to derived category of equivariant sheaves on "Spec L".
(moduli stack of 1-dim formal groups)
$X \mapsto \mathcal{O}^*_X$ Index $* \in \{0,1\}$ even/odd graded
$SHot$, $\otimes$-triangulated, , $(SHot)_{(compact)}$ compact objects (finite CW-complexes)
$\exists$ coh. functor $X \mapsto \mathcal{O}^*_X$ $SHot \to \mathbb{Z}/2\mathbb{Z}$-graded coherent sheaves of graded-comm algebras over $(Fgps)$?

Over $\mathbb{Q}$: pt/G_m, get $H^*(X;\mathbb{Q})$

No notion of cohomological functor with values in algebras ...
shifts in degree don't make sense
from $Ho(spaces) \to$ sheaves of algebras
but after suspension (in $SHot$) lose alg. structure
$Ho(Space) \to \mathbb{Z}/2$-gr. sheaves of gr. comm. algebras.

Morphisms in $(Fgps)$: <u>all</u> coord changes, not just strict isomorphisms.
Over $\mathbb{Q}$: $G_m \circlearrowleft$: sheaf on BG_m = graded vector space.
even, odd parts will both be G_m-rep's.

G_a in char p ! G_m action preserves law, get $\mathbb{Z}$-grading

Other theories $\mathbb{G}_m$ won't stabilize group law $\Rightarrow$ don't get $\mathbb{Z}$-grading e.g. $K(n)$ $\exists$ $2(p^n-1)$ cyclic grading
even/odd p^n roots of unity

Suspension of X $\Longleftrightarrow$ $\otimes \mathcal{O}^*_{S^2}$ line bundle defined by two-sphere $\mathcal{O}^*_{\Sigma X} = \mathcal{O}_X^{*+1}$

$\mathcal{O}_{\Sigma^2 X} \simeq \mathcal{O}_X \otimes \mathcal{O}_{S^2}$: non-equivariantly $\mathcal{O}_{\Sigma^2 X} \simeq \mathcal{O}_X$

Parameter corresponds to Chern generator: so $\mathcal{O}_{S^2}$ corresponds to tangent bundle of S^2 .... basic character of $\mathbb{G}_m$

For S^1: can't take square root ... maybe need not $\mathbb{G}_m$ but $\sqrt{\mathbb{G}_m}$ as ~~Super~~ extension, accounts for grading, $\mathbb{Z}/2 \in$ Center $\sqrt{\mathbb{G}_m}$.

$\rightsquigarrow$ Gerbe of square roots of basic line bundle ... are these symmetries of line acting or two-sheeted covering?

$\mathcal{O}_{S^1} = \mathcal{O}_{S^0}[e]/(e^2)$. $\mathcal{O}_{S^0} = \mathcal{O}$
for spaces (non reduced cohomology)
for spectra ...
$\mathcal{O}_{S^2}$ should be canonical bundle over (Fgps),
fiber is Lie algebra of formal group.
Should have square root of this bundle, $\mathcal{O}_{S^1}$.
In char 0 . $\mathbb{G}_m$, acts by standard character on tangent space
- need square root of this! i.e. double covering of diffeo $\supset$ center $\mathbb{Z}/2$.

Given stack & line bundle construct super stack via gerbe of square roots of bundle

$$1 \to \mathrm{Diff}_0 \to \widetilde{\mathrm{Diff}} \to \widetilde{\mathbb{G}_m} \to 1$$

$\mathbb{Z}[t_k \mid k \geq 1]$ $\quad \mathbb{Z}[t_0, t_0^{-1}]$
$$T \mapsto t_0^2 T + \sum_{k\geq 1} t_k T^{k+1}$$

Morava in Browder volume ... cobordism correspondences $Z \to X \times Y$ & their algebra, square root arises ...

$\mathrm{Ho}(\mathrm{Spec}) \longrightarrow$ Sheaves on superstack
Suspension $\longrightarrow$ $\otimes \mathcal{O}_{S^1}$
join $\longrightarrow$ tensor product modulo some Tor ...

Dual Hopf algebra $MU_* \otimes S^*$
- degrees get mixed up
Maps $(S^m, S^n) = 0$ if $m<n$
Alg-top intrinsically 1-dim! everything built from spheres ... $\rightsquigarrow$ 1-dim formal gps (Atiyah)

1-dim: $\mathbb{CP}^\infty = BS^1$ ...
Borel model of equivariant cohomology very "small"
· fiber at identity of putative bigger object on some group object ...

$X, Y = S^0$: $Ext(\mathcal{O}, (\text{canonical})^*)$
Line bundle is tangent to universal 1-dim formal group
m/m^2 cotangent. $0 \to (\text{cotan})^2 \to m/m^3 \to \underset{\text{cotangent}}{m/m^2} \to 0$
- splits over fixed point but not nec over moduli space ... $\to$ canonical
$Ext^1(\mathcal{O}(-2), \mathcal{O}(-1)) \leftrightarrow Ext^1(\mathcal{O}$ by switching$)$
... element of E_2^n term. Does it survive?

On big cell of stratification: localization of K-theory
— gives first line of spectral sequence ...
where image of J lies.
Elliptic coh gives new info only at primes 2, 3. (compared to K-theory)
gives everything up to dim 55
TMF agrees with classical modular forms away from 2, 3
Analog at higher primes comes from ht $p-1$ at p.
e.g. $p=3$ ht 2 contributes ...

Ell. curve $\to$ f. gps is formally smooth: but stabilizers on ell curve side smaller.
· Evidence with correlation: ht n for $X \iff$ ht $n-1$ for LX
$K(\text{loop space } X) \leftrightarrow$ ell coh (X)

$\underset{ht = \infty}{H^*_T}(LX) \sim \underset{ht=1}{KX}$ $\quad K_T LX \sim \underset{ht\ 2}{Ell}\ X$

have Line bundle which is square root of the Lie algebra...
- considering square evaluation : 1-dim odd vector space & 1-dim even, square goes odd → even ⟹ supercommutator.
Can we lift this to formal group level ?

<u>Adams-Novikov s.s.</u> $X \longmapsto \mathcal{O}_X^*$ contravariant.

"$E_2 = \mathrm{Ext}^*_{\text{stack}}(\mathcal{O}_Y, \mathcal{O}_X) \Longrightarrow Gr_* [X,Y]$ = Ext in SHot"

$[X,Y] = \lim_{l \to \infty} \mathrm{Map}[S^{k+l} X, S^l Y]$

not quite...
$[S^0, X] \Rightarrow H^*(\text{stack}, \mathcal{O}_X)$
On <u>homology</u> theory (covariant version)
$X = S^0 \quad [S^0, Y] = \pi_n^S(Y) \Leftarrow H^*(\text{stack}, \mathcal{O}_Y(n))$
two indexes : $\mathrm{Ext}^*(\mathcal{O}_Y, \mathcal{O}_X(n)) \Rightarrow [X,Y]$
Hopkins : stack of elliptic curves instead.

$f(x+y) = f(x)+f(y)$, formal autos are $f(x) = \sum a_i X^{p^i}$
Composition. $f(g(x)) = \sum f_i g_k^{p^i} X^{p^{i+k}}$
⇝ coproduct $\Delta f_i = \sum_{j+k=i} f_j \otimes f_k^{p^j}$

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Adams-Novikov : $E_2^{s,t} = \cancel{\mathrm{Ext}^{s,t}}$ $\mathrm{Ext}^s_{S\text{-sheaves}}(\mathcal{O}_X, \mathcal{O}_Y(t)) \Longrightarrow \mathrm{Ext}_{Ho(\text{Spaces})}(X,Y)$
$S = (1\text{-dim f.gps})$
tensor with canonical line bundle^t.

$t \in \mathbb{Z}$ .. X finite complex : bounded below.
"no absolute perspective" - can shift : twin

Exts here are just cohomology as ~~MU~~ ~~Landweber~~ Land-Novikov MU_*MU - module ......

Homology version : X finite cx $MU_*X = 0 \quad * < 0$
Landweber-Novikov coalgebra $LN = MU_*(pt) \otimes S_* \quad (= 0 \quad * < 0)$
S_* = fns on group of diffeos
$\mathrm{Ext}(MU_*(X), MU_*(X))$ - everything positively graded
LN-comodules

order 2 $\eta : S^3 \to S^2$ basic Hopf map — natural candidate
order 24 $\nu : S^7 \to S^3$ for Drinfeld extension class
(generates $\pi^{n+1}(S^n) = \mathbb{Z}/2$)

Any cohomology theory is a fiber functor for stable homotopy category ⟹ automatically enriched by its operations ⟹ Adams spectral sequence, trying to calculate maps in stable homotopy category from Ext groups over operations... won't converge to full thing for periodic theories: e.g. K theory just calculates bottom row ... H^* will but not efficiently..

G simple Lie group $\pi_3(G) \cong \mathbb{Z}$ ⟹
$G \to K(\mathbb{Z}, 3)$ E-M space
Fact: $E_8 \to K(\mathbb{Z}, 3)$ homotopy equivalence up to dim. 12!

$BU\langle 6\rangle$: 6-connected cover of BU ...
$\pi_* BU = 0 \underset{c_1}{\mathbb{Z}} 0 \underset{c_2}{\mathbb{Z}} 0 \mathbb{Z}$ etc so
$BU\langle 6\rangle$ just means killing off c_1, c_2

$MO\langle 8\rangle$: kill a Stiefel-Whitney w_3 : "String" manifolds free loop space is spin. Higher $MO\langle n\rangle$'s & $BU\langle n\rangle$'s have nasty big cohomology...