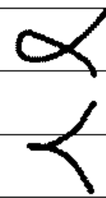
Mircea Mustata : FM on Cubic Curves

Note Title 5/15/2007

Setup:

E one of $\left\{\begin{array}{l}\text{ell curve} \\ \left.\begin{array}{l}\text{nodal} \\ \text{cuspidal}\end{array}\right\} \text{rat. curve}\end{array}\right.$



$\pi : \mathbb{P}^1 \to E$ normalization

$$\mathrm{Pic}^0(E) \xrightarrow{\sim} E_{sm} \qquad q \mapsto \mathcal{O}(q-b)$$

$$\cap$$

$\overline{\mathrm{Pic}^0}(E)$ = rank 1 t.f. sheaves of degree zero on E

$$\cong$$

$E = E_{sm} \cup \infty$ in singular cases

$\infty \longleftrightarrow \mathfrak{m}_\infty \otimes \mathcal{O}(b)$ twisted ideal sheaf of the singular point

$\exists$ Involution on E which on E_{sm} is $L \longmapsto L^{-1}$.

$\exists$ universal object $\mathcal{P}^\vee$, a torsion free sheaf on $E \times E$

$$\mathcal{P}^\vee = I_\Delta \otimes \left(\mathcal{O}(b) \boxtimes \mathcal{O}(b)\right)$$

(inverse to family $q \mapsto \mathcal{O}_{q-b}$)

X, Y smooth projective

$$Q \in D^b(\mathrm{Coh}\, Y \times X)$$

$$\begin{tikzcd} & Y \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y & & X \end{tikzcd}$$

$$\Phi = \Phi^Q_{Y \to X} : D^b(\mathrm{Coh}_Y) \to D^b(\mathrm{Coh}_X)$$

$$\Phi = R\pi_{2*}\left(Q \overset{L}{\otimes} \pi_1^*(-)\right)$$

Keep in mind: Y = moduli space of vector bundles on X, Q = universal object

$\Phi(\mathcal{O}_y) = \mathcal{O}_y$ vector bundle corresponding to y.

Grothendieck duality: have a left-adjoint functor

$$\Phi^{Q^\vee}_{X \to Y} \otimes \pi_2^* \omega_X[\dim X]$$

& a right adjoint $\Phi^{Q^\vee}_{X \to Y} \otimes \pi_1^* \omega_Y[\dim Y]$

where $Q^\vee = \underline{R\mathrm{Hom}}(Q, \mathcal{O}_{Y \times X})$

Theorem (Bondal-Orlov + Bridgeland)

Φ is an equivalence of categories $\iff$

1. $\mathrm{Hom}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)) \simeq k \quad \forall y \in Y$
2. $\mathrm{Ext}^i(\Phi(\mathcal{O}_{y_1}), \Phi(\mathcal{O}_{y_2})) = 0$ unless $y_1 = y_2$, $0 \leq i \leq \dim X$
3. $\dim X = \dim Y$ & $\Phi(\mathcal{O}_y) \otimes \omega_X \simeq \Phi(\mathcal{O}_y) \quad \forall y \in Y$

Back in our situation:

<u>Theorem</u> $\Phi^{\mathcal{P}^\vee}_{E \to E} : D^b(\mathrm{Coh}_E) \circlearrowleft$ is an equivalence of categories

eg if E is an elliptic curve, $L_1, L_2 \in \mathrm{Pic}^0 E$

$\mathrm{Ext}^i(L_1, L_2) \simeq H^i(L_2 \otimes L_1^{-1}) = 0$ for $L_1 \neq L_2$

k if $i = 0$ $L_1 \simeq L_2$.

- $\mathcal{P}^\vee$ is flat over E, so preserves bounded derived category.
- ω_E trivial (by adjunction for cubic plane curve)

- inverse is given by $\Phi^{\mathcal{P}}_{E\to E}[1]$.

  $\mathcal{P}$ also a sheaf & flat over E.

- using the fact that every bounded complex of quasicoherent sheaves is $\varinjlim$ such complexes with coherent cohomology

  $\rightsquigarrow$ get same statement $D^b(Qcoh_E)$ $\circlearrowleft$.

Example: M torsion free & semistable of deg <0

$\Rightarrow \mathrm{Hom}(\mathcal{O}_x, M\otimes\mathcal{L})=0 \quad \forall \mathcal{L}\in \mathrm{Pic}^0 X$

$\Rightarrow \Phi(M)[-1]$ is a sheaf

E elliptic, ∂ = unique up to scalar vector field

E cuspidal or nodal $\exists!\ \partial$ up to scalar which is invariant under the group E_{sm}

Nodal case: take $\partial = z\partial_z$ on $\mathbb{P}^1$

Cuspidal case: take ∂_z on $\mathbb{P}^1$.

$\mathcal{D}_{bg} \subset \mathcal{D}_E$ subring generated by $\mathcal{O}_E$, ∂.

– a special D-algebra

<u>Theorem</u> (Polishchuk-Rothstein)

If $\mathcal{A} = (p_{14})_* (p_{12}^* \gamma \otimes p_{23}^* \mathcal{D}_{bg} \otimes p_{34}^* \gamma^\vee)$

Then Φ induces an equivalence between

$$D^b(\mathcal{D}_{bg}\text{-mod}) \simeq D^b(\mathcal{A}\text{-mod})$$